Topics in Asset Pricing

Assignment 3: Pecuniary externality — To be submitted in pdf format to hombert@hec.fr) or in class before the beginning of the class on March 16th.

There are three date $t = 0, 1, 2$, one risky asset, and two types of agents: "arbitrageurs" and "investors", each in mass one. The asset is produced (issued) at date 0 at marginal cost 1 by arbitrageurs. A possible interpretation is that arbitrageurs are banks and the asset is loans made by the bank to firms and households. The supply of the asset is thus endogenous and is equal to arbitrageurs' investment in the asset at date 0, which is denoted by X_0^{arb}.

The asset makes a random payoff at date 2. The payoff is V_H with probability $1-q$ and V_L with probability q, where:

$$0 < V_L < 1 < V_H, \tag{1}$$

$$(1-q)V_H + qV_L > 1, \tag{2}$$

$$(1-q)\,(V_H - 1) - q\Big(1 + \frac{V_L}{k}\Big) < 0. \tag{3}$$

(Assumptions (2) and (3) will be useful for question 4.)

The asset payoff is realized at date 1. Then, still at date 1, the asset is traded among arbitrageurs and investors. For $\omega \in \{H, L\}$, when the date-2 asset payoff is V_ω, the asset price at date 1 is denoted by $P_{1,\omega}$ and the arbitrageurs' and investors' holdings of the asset at date 1 are denoted by $X_{1,\omega}^{arb}$ and $X_{1,\omega}^{inv}$, respectively.

Arbitrageurs' date 1-holdings of the asset are subject to the leverage constraint:

$$k|X_{1,\omega}^{arb}| \leq W_{1,\omega}^{arb}, \tag{4}$$

where $k > 0$ and $W_{1,\omega}^{arb}$ is arbitrageurs' date 1-wealth in state $\omega \in \{H, L\}$.

Investors face no leverage constraint, but they incur a cost $\frac{c}{2}(X_{1,\omega}^{inv})^2$ from holding the asset, where $c > 0$. An interpretation is that investors are not natural holders of the asset: when they buy it, they receive the payoff V_ω but incur a cost reflecting their comparative disadvantage at holding the asset.

There is also a risk-free asset with an exogenous rate of return normalized to 0. All agents are risk neutral, have a zero discount rate, and are competitive (they take prices as given.)

We assume that at date 1, in state $\omega = H$ arbitrageurs' leverage constraint (4) is slack and $P_{1,H} = V_H$, whereas in state $\omega = L$ the leverage constraint binds and $P_{1,L} < V_L$.

Question 1 Consider an arbitrageur who chose a position X_0^{arb} in the risky asset at date 0. Show that its demand for the risky asset at date 1 in state $\omega = L$ is:

$$X_{1,L}^{arb}(X_0^{arb}, P_{1,L}) = \frac{W_0 + (P_{1,L} - 1)X_0^{arb}}{k}. \tag{5}$$

Question 2 Show that investors' demand for the risky asset at date 1 in state $\omega = L$ is:

$$X_{1,L}^{inv}(P_{1,L}) = \frac{V_L - P_{1,L}}{c}. \tag{6}$$

Question 3 Consider that all arbitrageurs have chosen a position X_0^{arb} in the risky asset at date 0.

(a) Write the market clearing condition for the risky asset at date 1 in state $\omega = L$.

(b) Explain in words why the demand may be increasing or decreasing in $P_{1,L}$.

From now on, we assume that the parameters of the model are such that the demand is strictly decreasing in the price, which is equivalent to:

$$cX_0^{arb} < k. \tag{7}$$

(c) Show (e.g., graphically by plotting the demand and supply functions) that this assumption implies:

$$\frac{dP_{1,L}}{dX_0^{arb}} < 0. \tag{8}$$

In questions 1 to 3 you studied the equilibrium at date 1 taking as given arbitrageurs' decisions at date 0. In questions 4 and 5 you will analyze arbs' decisions at date 0. In question 4 you will calculate the decentralized equilibrium. In question 5 you will study whether the social planner can increase welfare relative to the decentralized equilibrium.

Question 4

(a) Determine the optimal investment in the risky asset X_0^{arb} of an arbitrageur at date 0 as a function of the anticipated price $P_{1,L}$. To answer this question it will be useful to define $P_{1,L}^*$ as the unique solution in $(0, V_L)$ to:

$$(1-q)(V_H - 1) - q(1 - P_{1,L})\Big(1 + \frac{V_L - P_{1,L}}{k}\Big) = 0. \tag{9}$$

(b) Your answer to question (a) implies that the equilibrium price of the risky asset at date 1 in state $\omega = L$ must be equal to $P_{1,L}^*$. What is the equilibrium value of X_0^{arb}?

Question 5 In this question we ask whether the equilibrium is constrained efficient. To answer this question, we consider a social planner who can choose arbitrageurs' investment in the risky asset at date 0. At date 1, the planner lets agents choose their privately optimal holdings of the asset as calculated in questions 1 and 2 and lets the market clear as analyzed in question 3.

We consider utilitarian welfare equal to the sum of utilities of all agents:

$$W = U^{arb}(X_0^{arb}, X_{1,L}^{arb}, P_{1,L}) + U^{inv}(X_{1,L}^{inv}, P_{1,L}),$$

where

$$U^{arb}(X_0^{arb}, X_{1,L}^{arb}, P_{1,L}) = (1-q)\Big[W_0 + \big(V_H - 1\big)X_0^{arb}\Big] + q\Big[W_0 + \big(P_{1,L} - 1\big)X_0^{arb} + \big(V_L - P_{1,L}\big)X_{1,L}^{arb}\Big]$$

and $X_{1,L}^{arb} = X_{1,L}^{arb}(X_0^{arb}, P_{1,L})$ is given by (5), and

$$U^{inv}(X_{1,L}^{inv}, P_{1,L}) = q\Big[\big(V_L - P_{1,L}\big)X_{1,L}^{inv} - \frac{c}{2}\big(X_{1,L}^{inv}\big)^2\Big]$$

and $X_{1,L}^{inv} = X_{1,L}^{inv}(P_{1,L})$ is given by (6), and $P_{1,L} = P_{1,L}(X_0^{arb})$ is determined by the market clearing condition you wrote in your answer to question 3(a).

We want to determine whether the social planner, starting from the decentralized equilibrium, can increase welfare by making a small change in X_0^{arb}. To answer this question, we compute the total derivative of W with respect to X_0^{arb}:

$$\frac{dW}{dX_0^{arb}} = \left(\frac{\partial U^{arb}}{\partial X_0^{arb}} + \frac{\partial U^{arb}}{\partial X_{1,L}^{arb}}\frac{\partial X_{1,L}^{arb}}{\partial X_0^{arb}}\right) + \frac{dP_{1,L}}{dX_0^{arb}}\left[\left(\frac{\partial U^{arb}}{\partial P_{1,L}} + \frac{\partial U^{inv}}{\partial P_{1,L}}\right) + \frac{\partial U^{arb}}{\partial X_{1,L}^{arb}}\frac{\partial X_{1,L}^{arb}}{\partial P_{1,L}} + \frac{\partial U^{inv}}{\partial X_{1,L}^{inv}}\frac{\partial X_{1,L}^{inv}}{\partial P_{1,L}}\right],$$

and we evaluate this derivative at the value of X_0^{arb} in the decentralized equilibrium.

(a) Which terms in this equation are equal to zero? [Answering this question requires almost no calculation: an explanation in a few words or with a short equation is sufficient for each one.]

(b) Show that $dW/dX_0^{arb} < 0$.

This implies that the social planner can improve welfare by reducing X_0^{arb} below its competitive level.

(c) Explain the intuition in words.