Topics in Asset Pricing

Assignment 2: Risk averse arbitrageurs — To be submitted in pdf format to hombert@hec.fr) or in class before the beginning of the class on March 9th.

There are two risky assets A and B in zero supply. A and B are traded at $t = 1$ and have random payoffs at $t = 2$. For $i = A, B$, we denote the price of asset i at $t = 1$ by P_i and the payoff at $t = 2$ by $V_i = V + \epsilon_i$, where $V > 0$ and ϵ_i is a random variable realized at $t = 2$. (ϵ_A, ϵ_B) is jointly normal:

$$(\epsilon_A, \epsilon_B) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix} \right), \qquad \sigma^2 > 0, \quad \rho \in [-1, 1].$$

There is also a risk-free asset with an exogenous rate of return normalized to 0.

There are three types of agents: A-investors, B-investors, and arbitrageurs. For $i = A, B$, i-investors can only trade asset i and the risk-free asset. They also receive an endowment shock at $t = 2$ that is perfectly correlated with asset i and is equal to $E_i \times V_i$, where E_i is a constant that can be positive or negative. Therefore, if i-investors take a position in X_i shares of asset i at $t = 1$, their wealth at $t = 2$ is equal to:

$$W_{2,i} = (E_i + X_i)(V + \epsilon_i) - P_i X_i.$$

i-investors have CARA preferences with risk aversion $\alpha > 0$ over wealth at $t = 2$:

$$u_i(W_{2,i}) = -e^{-\alpha W_{2,i}}.$$

Arbitrageurs can trade assets A and B and the risk-free asset. They have CARA utility with risky aversion $\gamma > 0$ over wealth at $t = 2$. There is a mass 1 of each type of agents and they are competitive (they take prices as given). The endowment shock at $t = 2$ can reflect income from labor or from other non-tradable assets.

The interpretation of this model is that assets A and B may be assets in two different countries, or in two different asset classes, or maybe two stocks in different industries, and A-investors and B-investors are specialized in one specific asset and do not trade the other asset. This pattern is sometimes called *market segmentation*.

When showing your answers to the following questions, you need to show the main equations that you solve to find the solutions, but you don't need to show all the steps of the calculation. When you are asked to "explain" a result, it means explaining the economic mechanism in words (it doesn't need to be long.)

Question 1 For $i = A, B$, show that the demand of i-investors for asset i at $t = 1$ is equal to:

$$X_i(P_i) = -E_i + \frac{V - P_i}{\alpha\sigma^2}. \tag{1}$$

Question 2 In the demand function (1), which term would you call the *hedging component* and which one would you call the *speculative component*? Explain the intuition for each one. One of these components depends on α and σ^2: explain briefly the intuition.

Note that E_i may also be interpreted as a demand shock: When $E_i > 0$, i-investors want to sell asset i for reasons unrelated to the expected payoff of asset i. This might be interpreted as a micro-foundation for the demand shocks studied in class.

Question 3 Show that the demand of arbitrageurs for assets A and B is:

$$Y_A(P_A, P_B) = \frac{V - P_A - \rho(V - P_B)}{\gamma(1-\rho^2)\sigma^2}, \tag{2}$$

$$Y_B(P_A, P_B) = \frac{V - P_B - \rho(V - P_A)}{\gamma(1-\rho^2)\sigma^2}. \tag{3}$$

Question 4 If $\rho > 0$, explain why the arbitrageurs' demand for asset B is increasing in P_A.

Question 5 If $\rho > 0$, $P_A < V$, and $P_B = V$, explain why the arbitrageurs' demand for asset A is increasing in ρ.

From now on, we assume the endowments of A-investors and B-investors are such that $E_A = -E_B > 0$. Following the interpretation of E_i as a demand shock, this case might describe a situation in which asset A is removed from a market index and asset B is added to the index.

Question 6 Show that the equilibrium prices are equal to:

$$P_A = V - \frac{\alpha\gamma(1-\rho)\sigma^2}{\alpha + \gamma(1-\rho)} E_A, \tag{4}$$

$$P_B = V - \frac{\alpha\gamma(1-\rho)\sigma^2}{\alpha + \gamma(1-\rho)} E_B. \tag{5}$$

Question 7 Explain why P_A is decreasing in E_A, but less so when ρ is close to 1.