Incentive Constrained Risk Sharing, Segmentation, and Asset Pricing*

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Abstract

Incentive problems make securities' payoffs imperfectly pledgeable. Introducing these problems in an otherwise canonical general equilibrium model yields a rich set of implications. Security markets are endogenously segmented. There is a basis going always in the same direction: the price of any risky security is lower than that of the replicating portfolio of Arrow securities. Equilibrium expected returns are concave in consumption betas, in line with empirical findings. As the dispersion of consumption betas of the risky securities increases, incentive constraints are relaxed and the basis reduced. When hit by adverse shocks, relatively risk tolerant agents sell their safest securities.

Keywords: General Equilibrium, Asset Pricing, Collateral Constraints, Endogenously Incomplete Markets.

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1 Introduction

Financial markets allow agents to share risk by trading contingent claims, such as futures, options or Credit Default Swaps. These contingent claims are promises to make payments in the future, if some states occur. To deliver on these promises, agents must have assets generating resources they can use to pay counterparties. We refer to these promise-backing assets as collateral. This is a broad definition of collateral, encompassing securities and real assets.

Collateral, however, is imperfectly pledgeable, because when an agent defaults it is costly and difficult for creditors to seize and realize the value of his assets. Because creditors want to avoid the cost of default, debtors can ex post renegotiate liabilities, down to the collateral value minus bankruptcy costs (as in Kiyotaki and Moore, 1997; Kiyotaki, 1998). Ex ante, this sets a limit to what can be credibly pledged. Imperfect collateral pledgeability limits the extent to which agents can credibly promise contingent payments, i.e., issue contingent claims, to share risk in financial markets.

Costs of default are significant for financial firms even when there are "safe harbor" provisions, as documented by Fleming and Sarkar (2014) and Jackson, Scott, Summe and Taylor (2011) in case studies of the bankruptcy of Lehman Brothers Holdings Inc. For instance, Fleming and Sarkar (2014, p. 193) write that "it has been alleged that Lehman (...) failed to segregate collateral", that creditors to these claims "were unable to make recovery through the close-out netting process and became unsecured creditor to the Lehman estate", and that "counterparties did not know when their collateral would be returned to them, nor did they know how much they would recover given the deliberateness and unpredictability of the bankruptcy process."

The goal of this paper is to study the implications of imperfect pledgeability for risk sharing and asset pricing. To conduct this analysis, we consider a canonical one-period general equilibrium model with a single consumption good. At time 0, competitive risk averse agents are endowed with shares of securities in positive aggregate supply (or "Lucas trees") generating consumption good output at time 1. Lucas trees are risky and heterogenous: time-1 output varies across states of nature and across trees. The agents can take long and short positions in these trees and while doing so buy or sell the whole vector of outputs generated by a tree in all the possible states of nature. Agents, however, can also take long and short positions in a complete set of Arrow securities in zero net supply. Any long security positions, either in trees or in Arrow securities, can be used as collateral.

If collateral was perfectly pledgeable, the first best would be attained in equilibrium. Agents would use the output from their collateral to make the payments they promised. This would enable agents to achieve efficient insurance. In this complete market consumption-CAPM world, only the risk associated with aggregate output would be priced. Finally, agents would be indifferent between holding a tree and a corresponding replicating portfolio of Arrow securities, since both would have the same price and the same payoff. As a result, the allocation of trees would be indeterminate.

In contrast, we analyze equilibrium with imperfectly pledgeable collateral. Incentive compatibility implies that an agent's pledgeable income in a given state is only a fraction of his long positions' payoff in that state. This limits the state contingent transfers agents can promise one another and hence constrains the provision of insurance. Thus, even though agents can trade a complete set of Arrow securities, and even if the span of tree payoff is complete, risk sharing can be imperfect and the market endogenously incomplete. We show in this context that equilibrium exists and is constrained Pareto optimal (constrained optimality of equilibrium obtains because, in a one-period model, prices do not appear in incentive compatibility constraints.)

Equilibrium risk sharing is imperfect only if the first best cannot be achieved with only long positions in trees. That is, risk sharing is imperfect only if agents must establish short positions to share risk. Without short positions, i.e., without liabilities, imperfect pledgeability is not an issue. Because short positions create incentive problems, an equilibrium allocation, which is constrained Pareto optimal, minimises the issuance of Arrow securities. When short positions arise in equilibrium, they concern Arrow securities, not trees. This is because shorting trees creates unnecessary liabilities (and therefore tighten incentive constraints), which selling Arrow securities can avoid. Thus, in equilibrium, agents hold long positions in trees generating payoffs as close as possible to their desired state-contingent consumption profile. Correspondingly different agents, with different desired consumption profiles, hold different trees. Equivalently, different trees are held by different clienteles, i.e., the market is endogenously segmented. Segmentation arises even though there is a complete set of Arrow securities, because of endogenous market incompleteness due to incentive constraints.

Incentive constraints also affect asset pricing: The price of a tree is equal to the value of its output evaluated at Arrow securities prices, minus the shadow incentive cost of the agent holding it. This implies there is a basis between the price of a tree and that of the replicating portfolios of Arrow securities. Furthermore, the basis always goes in the same direction, as each tree is priced *below* its replicating portfolio of Arrow securities. This is striking because we assume trees and Arrow securities are equally pledgeable.

The basis is a deviation from the Law of One Price but does not constitute an arbitrage opportunity. In order to conduct an arbitrage trade, an agent would need to sell Arrow securities and use the proceeds to buy the tree. This is precluded by the incentive constraint: if the agent sold these Arrow securities, this would increase his liabilities, leading to a violation of his incentive constraint.

Another way to grasp the intuition for the basis is to compare the ease with which the payoffs of trees and portfolios of Arrow securities can be stripped: To strip the tree payoffs one needs to issue Arrow securities collateralized by the tree, which is costly in terms of incentives. In contrast, the payoffs of a portfolio of long positions in Arrow securities can be stripped, by selling some of these securities, without any incentive cost. Now the ease to strip payoffs is valued in the market because it helps agents structuring portfolios that fit their desired consumption profiles. Hence a tree is valued less than its replicating portfolio of Arrow securities.

Following the same logic, there is a basis between the price of a tree and that of a replicating portfolio of long positions in trees and Arrow securities. For example our model predicts, in line with empirical evidence, that the price of a convertible bond is lower than that of the replicating portfolio of the straight bond and the call on the issuer's stock. Likewise, in a binomial-tree version of our model, the price of a portfolio made up of a call option and a bond is greater than that of the underlying asset. This is in line with empirical evidence on option pricing.[1]

Moreover, equilibrium expected excess returns reflect two premia. The first premium increases in the covariance between the tree payoff and the consumption of an unconstrained agent, whose identity varies across states. The second premium increases with the covariance between the tree payoff and the shadow price of the incentive compatibility constraint of the agents holding it.

To further illustrate equilibrium properties, we consider the simple case in which there are two states,

[1]For example this observation corresponds to the empirical finding of Longstaff (1995) who writes that: "because an option can be viewed as a levered position in the underlying asset, the results suggests that it is more expensive to purchase the stock via the option market than in the stock market." See also Rubinstein (1994) and Bates (2000), among many others.

two agents' types with different relative risk aversion, and an arbitrary distribution of trees. In equilibrium, as in the first best (but to a lower extent), the consumption share of the more risk tolerant agent is smaller in the bad state than in the good state. That is, the more risk tolerant agent offers some insurance to the more risk averse agent. To implement this allocation, the more risk tolerant agent sells (resp. buys) Arrow securities that pay in the bad (resp. good) state. Hence his incentive compatibility constraint binds in the bad state and is slack in the good state. To mitigate incentive problems, the more risk averse agent holds trees with relatively large payoff in the bad state, so that he needs to purchase less Arrow securities from the risk tolerant agent. That is, the more risk averse agent holds trees with low consumption beta, while, by market clearing, the more risk tolerant agent holds trees with high consumption beta. This is a deviation from the two-fund separation principle, which illustrates how segmentation arises in equilibrium in our model. In this simple case a rich set of implications obtains.

First, expected excess returns are concave in consumption betas, in line with Black (1972) and recent evidence by Frazzini and Pedersen (2014) and Hong and Sraer (2016). That is, an intermediate beta tree is valued less than the replicating combination of a high and low beta tree, which illustrates the above stated general principle that a tree is valued less than a replicating portfolios of long positions in trees.

Second, holding aggregate risk and aggregate pledgeable income constant, the distribution of aggregate output across trees in each state matters for equilibrium outcomes. Namely, incentive constraints are less likely to impact equilibrium if the distribution of security betas is more dispersed. This is because the availability of low and high beta trees increases the ease with which agents can construct portfolios of trees with payoffs close to their desired consumption profiles. In practice, the dispersion of security betas might be lower when many firms are conglomerates; when the productive sector relies on a relatively small number of technologies; or when many corporations are financially distressed so that safe assets are scarce.

Third, our analysis sheds light on the consequences of shocks worsening incentive problems. Suppose the more risk tolerant agent is subject to a negative shock, increasing his shadow cost of holding assets. The agent sells his least risky holdings, for which his comparative advantage is the lowest. At the same time, the basis increases for all the assets initially held by the agent, so that there is comovement among these assets.

Literature: Our model is in line with the limited commitment literature, see Kehoe and Levine (1993, 2001), Holmström and Tirole (2001), Alvarez and Jermann (2000), Chien and Lustig (2009) and Gottardi and Kubler (2015). In these papers, and also in ours, while there is a complete set of Arrow securities, incentive constraints prevent full risk-sharing. However, these papers assume that some income (interpreted either as labour or corporate income) cannot be pledged nor traded, but tradeable assets can be pledged. In contrast, we assume that all income is generated by assets that can be traded and imperfectly pledged.[2] This difference in assumption generates a difference in results: In the limited commitment literature, the Law of One Price holds.[3] In contrast, our model exhibits equilibrium deviations from the Law of One Price.

Deviations from the Law of One Price have been obtained by another important strand of literature, in particular Hindy and Huang (1995), Aiyagari and Gertler (1999), Gromb and Vayanos (2002, 2017), Coen-Pirani (2005), Fostel and Geanakoplos (2008), Gârleanu and Pedersen (2011), Geanakoplos and Zame (2014), and Brumm, Grill, Kubler and Schmedders (2015). That literature differs from our paper and from the limited commitment literature in that it specifies plausible but exogenous financial constraints. In contrast, in our paper and in the limited commitment literature, financial constraints stem from the incentive compatibility condition that the agent must prefer to hold his promises rather than deviating.[4] Thus, the equilibrium allocation can be interpreted as the outcome of an optimal contracting process. Endogenizing constraints, moreover, yields new results on segmentation, bases, and equilibrium returns.

Fostel and Geanakoplos (2008), Geanakoplos and Zame (2014), Brumm, Grill, Kubler and Schmedders (2015), Geerolf (2015), Gromb and Vayanos (2002, 2017), and Lenel (2017) also analyze general equilibrium under collateral constraints. In that literature, each financial promise must be backed by its own collateral.[5] In our framework, by contrast, the constraint applies to the portfolio of assets and Arrow securities of an agent, in line with the practice of portfolio margining.[6] As mentioned above, a key finding in our framework

[2]Rampini and Vishwanathan (2017) also study risk-sharing under financial constraints (between a risk averse hedger and a risk neutral intermediary, via state-contingent debt). Their analysis, which does not consider the trading of assets, differs from ours, which focuses on pricing the cross section of assets.

[3]For example, Alvarez and Jermann (2000) write (on page 776): "The price of an arbitrary asset is calculated by adding up the prices of the corresponding contingent claims." Likewise, in Holmström and Tirole (2001), there is a single vector of state prices that is used to calculate the value of all claims, either corporate or non-corporate (see equation (20) and (23) on page 1849).

[4]In this context payoffs in case of deviation are explicitly specified. For example in Alvarez and Jermann (2000) agents must revert to autarky, while in Chien and Lustig (2009) the agents' holdings of a Lucas tree are seized, and in our model a fraction of the output of all long security positions held by the agent is seized.

[5]So the same asset, generating strictly positive output in two states, cannot be used to collateralize the issuance of two Arrow securities, promising payments in these two states.

[6]For example, on `http://www.cboe.com/products/portfolio-margining-rules`, one can read: "The portfolio margining

is that assets trade at a discount relative to replicating portfolios of derivatives. This could seem to contradict the "collateral premium" obtained by these papers. Similarly, this could seem to contradict the "liquidity premium" derived by the new monetarist literature for assets that can be used as means of payment (see, for example Lagos, 2010; Li, Rocheteau and Weill, 2012; Lester, Postlewaite and Wright, 2012; Venkateswaran and Wright, 2013; Jacquet, 2015). There is no contradiction, however: it is simply that the benchmark valuation is not the same for the premium and the basis results. The *premium*, which is the difference between the price of the asset and its counterfactual value based on the marginal utility of the agent holding it, obtains in these papers as well as in ours. Analyzing the *discount* at which real assets trade relative to replicating portfolios of Arrow securities is a contribution of our paper relative to that literature.

Krishnamurthy (2003) also studies general equilibrium under collateral constraints. Our paper is related with Krishnamurthy (2003) because (in both) collateral constraints limit hedging. However, our focus on asset pricing issues, such as risk premia, segmentation and deviation from the Law of One Price, differs from his focus on amplification mechanisms in production economies.

Finally, another related strand of the literature has proposed models of clientele, in which assets are held and priced by the endogenous segment of agents who value them most. This includes, for example, Amihud and Mendelson (1986), Dybvig and Ross (1986) and Sharpe (1991). We make two contributions relative to this earlier work. First, clienteles arise from on a new, incentive-based, economic mechanism. Second, we generate clientele without imposing short-selling constraints on individual assets.

The next section presents our model. Section 3 and 4 present general results on equilibrium. Section 5 presents more specific results, obtained when there are only two states and two types of agents. Proofs are in the online appendix.

rules have the effect of aligning the amount of margin money ... to the risk of the portfolio as a whole, calculated through simulating market moves up and down, and accounting for offsets between and among all products held..."

2 Model

2.1 Agents

There are two dates $t = 0, 1$. The state of the world ω realizes at $t = 1$ and is drawn from some finite set Ω according to the probability distribution $\{\pi(\omega)\}_{\omega\in\Omega}$, where $\pi(\omega) > 0$ for all ω. The economy is populated by finitely many types of agents, indexed by $i \in I$. The measure of type $i \in I$ agents is normalized to one. Agents of type $i \in I$ have Von Neumann Mortgenstern utility

$$U_i(c_i) \equiv \sum_{\omega\in\Omega} \pi(\omega) u_i\left[c_i(\omega)\right]$$

over time $t = 1$ state-contingent consumption, $c_i \equiv \{c_i(\omega)\}_{\omega\in\Omega}$. We take the utility function to be either linear, $u_i(c) = c$, or strictly increasing, strictly concave, and twice-continuously differentiable over $c \in (0, \infty)$. Without loss of generality, we apply an affine transformation to the utility function $u_i(c)$ so that it satisfies either $u_i(0) = 0$; or $u_i(0) = -\infty$ and $u_i(\infty) = +\infty$; or $u_i(0) = -\infty$ and $u_i(\infty) = 0$. In addition, if $u_i(0) = -\infty$, we assume that the agent's relative risk aversion remains bounded near zero: specifically, that there exists some $\gamma_i > 1$ such that, for all c small enough, $\frac{u_i'(c)c}{|u_i(c)|} \leq (\gamma_i - 1)$.[7]

2.2 Securities

There are two types of securities: trees in positive aggregate supply, and a complete set of Arrow securities in zero net supply.

Trees provide all real resources of the economy. The set of tree types is taken to be a compact interval that we normalize to be $[0, 1]$, endowed with its Borel σ-algebra. The distribution of tree supplies is a positive and finite measure $\bar{N}$ over the set $[0, 1]$ of tree types. We place no restriction on $\bar{N}$: it can be discrete, continuous, or a mixture of both. The payoff of tree $j \in [0, 1]$ in state $\omega \in \Omega$ is denoted by $d_j(\omega) \geq 0$, with at least one strict inequality in some state $\omega \in \Omega$. A technical condition for our existence proof is that, for all $\omega \in \Omega$,

[7] This implies the Constant Relative Risk Aversion (CRRA) bound $0 \geq u_i(c) \geq Kc^{1-\gamma_i}$ for all c small enough and some negative constant K. We use this technical condition in our existence proof to show that the maximum correspondence of the social planner's problem has a weakly closed graph and that, at points where some of the welfare weights are equal to zero, the maximized social welfare function is continuous in welfare weights. See the proof of Proposition C.1 in the Supplementary Appendix.

$j \mapsto d_j(\omega)$ is continuous. Economically, continuity means that trees are finely differentiated: nearby trees in $[0,1]$ have nearby characteristics. Continuity is a mild assumption since we do not impose any restriction on the distribution $\bar{N}$ of supplies.[8] As will become clear later, we consider a continuum of trees for two reasons. First, it will demonstrate clearly that our results do not arise because the set of tree payoffs is in some way incomplete. Second, in Section 5, it will make it easier to explicitly characterize patterns of segmentation.

Agent $i \in I$ is initially endowed a strictly positive share, $\bar{n}_i > 0$, in the market portfolio of trees, $\bar{N}$. Agents' shares in the market portfolio add up to one, $\sum_{i \in I} \bar{n}_i = 1$. Agents have no initial endowment of Arrow securities.

At time zero, an agent can take long and short positions in all assets, trees and Arrow securities. The portfolio of agent's i tree positions is denoted by $N_i \equiv (N_i^+, N_i^-)$, where N_i^+ is the portfolio of long positions, and N_i^- is the portfolio of short positions. Both N_i^+ and N_i^- belong to the set $\mathcal{M}_+$ of positive and finite measures over the set of tree types, $[0,1]$. Likewise, the portfolio of Arrow security positions, long and short, is denoted by $a_i \equiv \{a_i^+(\omega), a_i^-(\omega)\}_{\omega \in \Omega} \in \mathbb{R}_+^{2|\Omega|}$.

2.3 Incentive Constraints

At time one, an agent can strategically default on the contractual obligations created by short positions in trees and Arrow securities. As discussed in the introduction in the Lehman Brother's case, when counterparties default, collateral recovery is costly and imperfect. Debtors can take advantage of such imperfect recoverability to renegotiate liabilities. This opportunity to renegotiate limits the repayments agents can credibly promise to make.

To capture this process in the simplest possible way, we assume an agent can make a take-it-or-leave-it offer to his creditors, who, if they refuse, can only recover a fraction $1-\delta \in (0,1]$ of the agent's long positions in trees and Arrow securities. So the agent can always renegotiate his liabilities down to a fraction $1-\delta$ of his total long positions. Therefore, this is the maximum amount he can credibly promise to repay, i.e., his

[8] For example, our model nests a standard specification with finitely many trees $k \in \{1, \ldots, K\}$, with state-contingent payoff $D^{(k)}(\omega)$ and positive supplies $S^{(k)} > 0$. Indeed, this discrete specification obtains by fixing a finite sequence $j_1 < j_2, \ldots < j_K$, choosing any continuous function $j \mapsto d_j(\omega)$ such that $d_{j_k}(\omega) = D^{(k)}(\omega)$ for $k \in \{1, \ldots, K\}$, and letting $\bar{N}$ be a discrete distribution with atoms at $j_1 < j_2, \ldots < j_K$, $\bar{N}_{j_k} - \bar{N}_{j_k -} = S^{(k)}$.

pledgeable income.[9] Correspondingly, we impose the following incentive compatibility constraint

$$\int d_j(\omega)\, dN_{ij}^- + a_i^-(\omega) \leq (1-\delta)\left[\int d_j(\omega)\, dN_{ij}^+ + a_i^+(\omega)\right], \tag{1}$$

for each $(i,\omega) \in I \times \Omega$, and where integrals are taken over the set of tree types, $j \in [0,1]$.[10] As shown by equation (1), the state-contingent payoff of all long positions serves as collateral for the state-contingent liabilities created by short positions. But the maximum liability the agent can credibly promise to repay, i.e., the pledgeable income, is lower than the face value of the collateral. The wedge between the two can be interpreted as a haircut and is increasing in δ. Haircuts are not imposed on an individual security basis, but at the level of the aggregate position, or portfolio of the agent. This is in line with the practice of "portfolio margining."[11]

2.4 Definition of Equilibrium

A *price system* for trees and Arrow securities is a pair (p,q), where $p : j \mapsto p_j$ is a continuous and strictly positive function for the price of tree j and $q = \{q(\omega)\}_{\omega\in\Omega}$ is a strictly positive vector in $\mathbb{R}^{|\Omega|}$ for the prices of Arrow securities.[12]

Given the price system (p,q), the time-zero budget constraint for agent i is:

$$\int p_j\ \left[dN_{ij}^+ - dN_{ij}^-\right] + \sum_{\omega\in\Omega} q(\omega)\left[a_i^+(\omega) - a_i^-(\omega)\right] \leq \bar{n}_i \int p_j\, d\bar{N}_j, \tag{2}$$

where the integrals are taken over the set of tree types, $j \in [0,1]$. At time one, agent i's consumption must

[9]Since there is a full set of Arrow securities, promising more than this maximum and subsequently defaulting would not complete the market and expand the agent's consumption possibilities. So, unlike in Geanakoplos and Zame (2014), there is no default on the equilibrium path.

[10]Equation (1) is a renegotiation-proofness condition: all agents anticipate that an allocation that would not satisfy (1) would be renegotiated to an allocation satisfying (1). See Appendix A for an explicit derivation.

[11]In the main body of the paper we assume for simplicity that δ is constant across agents and securities. As shown in Appendix C, our proof of existence and our asset pricing results generalize to the case in which δ depends on the identity i of the agent and of the type j of the security.

[12]Restricting attention to strictly positive prices is clearly without loss of generality: indeed, non-positive prices cannot be the basis of an equilibrium because they would result in infinite demand. We also assume that the price functional admits a natural dot-product representation based on a continuous function, p_j, of tree type. This entails some loss of generality, as the price functional should in principle be some arbitrary continuous linear functional, which may not have a dot-product representation. However, we show in Proposition 3, that there always exists an equilibrium in which the price functional admits this representation.

satisfy:

$$c_i(\omega) = \int d_j(\omega)\ \left[dN_{ij}^+ - dN_{ij}^-\right] + a_i^+(\omega) - a_i^-(\omega). \tag{3}$$

The problem of agent i is to maximize $U_i(c_i)$ with respect to a plan for consumption, $c_i = \{c_i(\omega)\}_{\omega\in\Omega}$, long and short portfolios of trees, $N_i = (N_i^+, N_i^-)$, and of Arrow securities, $a_i = \{a_i^+(\omega), a_i^-(\omega)\}_{\omega\in\Omega}$, subject to the incentive constraints (1) for all $\omega \in \Omega$, and to the time-zero and time-one budget constraints (2) and (3). An allocation is a collection of plans $(c_i, N_i, a_i)_{i\in I}$ for all agents. A *security market equilibrium* is, then, an allocation, $(c_i, N_i, a_i)_{i\in I}$, and a price system (p, q), such that for all $i \in I$, (c_i, N_i, a_i) solves agent's i problem given prices and all asset markets clear, that is:

$$\sum_{i\in I} N_i^+ = \bar{N}_i + \sum_{i\in I} N_i^- \tag{4}$$

$$\sum_{i\in I} a_i^+(\omega) = \sum_{i\in I} a_i^-(\omega) \text{ for all } \omega \in \Omega. \tag{5}$$

Our formulation of the agent's problem clearly shows that, since there is a complete set of Arrow securities, an agent can always avoid any incentive problems by selling all of his tree endowment, and only purchase a portfolio of Arrow securities corresponding to his desired state-contingent consumption profile. But this, of course, cannot be part of an equilibrium: if all agents sold all of their trees, the market clearing condition for trees, (4), would not hold. This simple observation intuitively implies our result about bases: for the tree market to clear, the price of trees will have to fall below that of Arrow securities. This also highlights that, in our model, binding incentive compatibility constraints is ultimately a general equilibrium phenomenon. This is in contrast with earlier models in which non-pledgeable income is not tradeable: in these environments, binding incentive constraints would already arise in partial equilibrium contract-theoretic settings.

3 Ruling Out Short Positions in Trees

In this section, we show that, in a security market equilibrium, two types of positions are weakly suboptimal: short tree positions, $N_{ij}^-([0,1]) > 0$, and simultaneous non netted long and short positions in the same Arrow security, $a_i^+(\omega) > 0$ and $a_i^-(\omega) > 0$. Because these positions are weakly suboptimal, equilibrium consumption and prices are not affected by whether they are allowed or not. In that sense they are irrelevant. This result allows us to define an equilibrium concept in the spirit of the standard Arrow-Debreu equilibrium (see Mas-Colell, Whinston and Green, 1995, page 691), ruling out these irrelevant asset positions, and thus imposing simpler budget and incentive constraints, but leading to identical equilibrium consumptions and prices.

Irrelevance of short tree positions, and of non netted positions. Our first result is that any equilibrium consumptions and prices can be supported without short positions in trees and non netted positions.

Lemma 1 *Consider any security market equilibrium* $(c_i, N_i, a_i)_{i\in I}$ *and* (p, q). *Then, there exists another security market equilibrium* $(\hat{c}_i, \hat{N}_i, \hat{a}_i,)_{i\in I}$ *and* $(\hat{p}, \hat{q})$ *such that:* $\hat{c}_i = c_i$, $\hat{N}_i^- = 0$, $\hat{a}_i^+(\omega)\hat{a}_i^-(\omega) = 0$ *for all* $\omega \in \Omega$, $\hat{p} = p$, *and* $\hat{q} = q$.

In the standard perfect and complete market model, Lemma 1 is immediate because agents' constraints and payoffs only depend on net security positions, and because the Law of One Price holds for all securities. Neither condition holds in the presence of incentive constraints, which makes the result non obvious.

To prove Lemma 1, we first construct the candidate equilibrium allocation $(\hat{c}_i, \hat{N}_i, \hat{a}_i)_{i\in I}$ by keeping consumption the same, $\hat{c}_i = c_i$, and by substituting and netting asset positions in (N_i, a_i). Namely, we substitute all short tree positions with short positions in replicating portfolios of Arrow securities. To restore market clearing for trees whose short positions have been eliminated, we scale down agents long positions in those trees and substitute them with long positions in replicating portfolios of Arrow securities. Finally, we net all long and short Arrow positions. Next, we show that, given prices (p, q), the candidate plan $(\hat{c}_i, \hat{N}_i, \hat{a}_i)$ remains optimal for each agent i. Since $\hat{c}_i = c_i$, all we need to show is that the candidate plan satisfies the incentive and budget constraints.

The incentive constraints hold for two reasons. First, substitution of tree positions by corresponding positions in replicating portfolio of Arrow securities does not impact incentive constraints, since the latter only depend on the total value of assets or liabilities, and not on their compositions. Second, netting positions in Arrow securities relaxes incentive constraints, since it reduces the total value of liabilities, on the left-hand side of (1), by more than the total pledgeable value of assets, on the right-hand side.

To see why the budget constraint holds, note that although the Law of One Price may not apply to all assets (as will become clear later), it must apply to all trees that, in our construction, are substituted by replicating portfolio of Arrow securities. Indeed, trees are only substituted if they are held short by some agents and long by others. But short tree positions can only be optimal for trees priced weakly above their replicating portfolio, and vice versa for long positions. Taken together, this implies that the Law of One Price must hold for all trees that are held both long and short, and therefore for all trees that, in our construction, are substituted by replicating portfolios.

An equivalent Arrow-Debreu equilibrium concept. Lemma 1 shows that we can assume without loss of generality that agents do not short trees and do not take simultaneous long and short positions in Arrow securities, $a_i^+(\omega)a_i^-(\omega) = 0$. This implies that, in the incentive compatibility constraint, agents never use long positions in Arrow securities as collateral for a short position in the same Arrow security. Hence, for incentive compatibility, it is necessary and sufficient that agents have enough tree collateral to secure their *net* Arrow security positions: $-\left[a_i^+(\omega) - a_i^-(\omega)\right] \leq (1-\delta) \int d_j(\omega)\, dN_{ij}^+$. Expressing $a_i^+(\omega) - a_i^-(\omega)$ in terms of consumption and tree payoff, we obtain the equivalent incentive constraint:

$$c_i(\omega) \geq \delta \int d_j(\omega)\, dN_{ij}^+ \text{ for all } \omega \in \Omega. \tag{6}$$

Next, we make the standard observation that the two sequential budget constraints, (2) and (3), can be consolidated in one single inter-temporal budget constraint:

$$\sum_{\omega\in\Omega} q(\omega)c_i(\omega) + \int p_j\, dN_{ij} \leq \bar{n}_i \int p_j\, d\bar{N}_j + \sum_{\omega\in\Omega} q(\omega) \int d_j(\omega)\, dN_{ij}^+. \tag{7}$$

These observations allow us to define an equivalent concept of *Arrow-Debreu equilibrium*, in which an agent purchases directly state-contingent consumption at time zero, without needing to consider explicitly positions in Arrow securities. Namely, *the problem of agent i* is to maximize $U_i(c_i)$ with respect to a plan for consumption and long tree positions, $(c_i, N_i^+) \in \mathbb{R}_+^{|\Omega|} \times \mathcal{M}_+$, subject to the intertemporal budget constraint (7) and the incentive constraints (6). An *allocation* is a collection $(c_i, N_i^+)_{i\in I}$ of consumption plans and long tree positions for every agent $i \in I$. An *Arrow-Debreu equilibrium* is an allocation $(c_i, N_i^+)_{i\in I}$ and a price system (p, q) such that, for all $i \in I$, (c_i, N_i^+) solves agent's i problem given prices and markets clear:

$$\sum_{i\in I} N_i^+ = \bar{N} \tag{8}$$

$$\sum_{i\in I} c_i(\omega) \leq \sum_{i\in I} \int d_j(\omega)\, dN_{ij}^+ \text{ for all } \omega \in \Omega. \tag{9}$$

We can now state that the concept of Arrow-Debreu equilibrium is equivalent to that of security market equilibrium in the following sense:

Proposition 1 *If (c_i, N_i, a_i) and (p, q) is a security market equilibrium, then there exists an Arrow-Debreu equilibrium $(\hat{c}_i, \hat{N}_i^+)$ and $(\hat{p}, \hat{q})$ such that $\hat{c}_i = c_i$, $\hat{p} = p$ and $\hat{q} = q$. Conversely, if $(\hat{c}_i, \hat{N}_i^+)$ and $(\hat{p}, \hat{q})$ is an Arrow-Debreu equilibrium, then there exists a security market equilibrium, (c_i, N_i, a_i) and (p, q), such that $c_i = \hat{c}_i$, $N_i^+ = \hat{N}_i^+$, $N_i^- = 0$, $p = \hat{p}$, $\bar{N}$-almost everywhere, and $q = \hat{q}$.*

That for each security market equilibrium, there exists an Arrow-Debreu equilibrium with the same consumption and prices follows from Lemma 1. The key reason why the converse holds, is that no agent strictly prefers to move away from the Arrow-Debreu allocation by taking short tree positions or non netted positions in Arrow securities, since, as in Lemma 1, such positions are dominated.[13]

4 Equilibrium analysis

Based on the equivalence result of Proposition 1, we focus, for the remainder of the paper, on Arrow-Debreu equilibria.

4.1 Incentive-Constrained Pareto Optimality and Existence

An allocation $(c_i, N_i^+)_{i\in I}$ is said to be *incentive-feasible* if it satisfies the incentive compatibility constraints (6) for all $(i, \omega) \in I \times \Omega$, and the feasibility constraints (8) and (9). An incentive-feasible allocation $(\hat{c}_i, \hat{N}_i^+)_{i\in I}$

[13]In the statement of the converse, the equality among prices is said to hold almost everywhere because it holds only for trees in strictly positive supply. Indeed, for securities in zero net supply, the Arrow-Debreu equilibrium does not rule out very high prices (since short sales are not allowed), while in the security market equilibrium such prices cannot arise (otherwise all investors would want to short sell the trees).

Pareto dominates the incentive-feasible allocation (c, N) if $U_i(\hat{c}_i) \geq U_i(c_i)$ for all $i \in I$, with at least one strict inequality for some $i \in I$. An allocation is *incentive-constrained Pareto optimal* if it is incentive-feasible and not Pareto dominated by any other incentive-feasible allocation. In our model, we have:

Proposition 2 *Any Arrow-Debreu equilibrium allocation is incentive-constrained Pareto optimal.*

The reason why optimality obtains in spite of incentive constraints is because prices do not show up in the incentive compatibility condition, so that there are no "contractual externalities".[14] See Prescott and Townsend (1984) and Kehoe and Levine (1993) for other examples of economies in which equilibrium is constrained optimal, and Gottardi and Kubler (2015) for examples in which it is not. Because there are no contractual externalities, the proof of Proposition 2 is similar to its perfect market counterpart: if an equilibrium allocation were Pareto dominated by another incentive feasible allocation, the latter must lie outside the agents' budget set. Adding up across agents leads to a contradiction.

As is standard, the Pareto efficiency result of Proposition 2 is especially useful because it facilitates the proof of equilibrium existence. Namely, following standard steps, we consider the problem of a Planner who assigns Pareto weights $\alpha_i \geq 0$ to each agent $i \in I$, with $\sum_{i \in I} \alpha_i = 1$, and chooses incentive feasible allocations to maximize the social welfare function, $\sum_{i \in I} \alpha_i U_i(c_i)$. The proof of existence then boils down to the problem of finding Pareto weights such that, given agents' initial endowment, the Planner's solution can be implemented in a competitive equilibrium without making any wealth transfers between agents. This leads to:

Proposition 3 *There exists an Arrow-Debreu equilibrium,* $(c_i, N_i^+)_{i \in I}$ *and* (p, q)*.*

The proof follows arguments found in Negishi (1960), Magill (1981), and Mas-Colell and Zame (1991) with some differences. First, our planner is now subject to incentive compatibility constraints. Second, the commodity space includes portfolios of trees and is therefore infinite dimensional, which makes it harder to apply separation theorems. Proposition 3 addresses these difficulties by explicitly deriving first-order necessary and sufficient conditions for the Planner's problem, and using the associated Lagrange multipliers to construct equilibrium prices.

[14] While Proposition 2 states that equilibrium is incentive constrained Pareto optimal, we show in Appendix C.3 that equilibrium is unconstrained Pareto optimal when it can be implemented without short positions. Without short positions, incentive compatibility conditions are slack.

While we cannot prove uniqueness in general, we show in Appendix C.4 that uniqueness obtains in a particular case of interest: when there are two types of agents with CRRA utilities, with relative risk aversion less than one.

4.2 First-order conditions

The key implications of our model are obtained based on agents' first-order conditions. Since agents have concave objectives and are subject to finitely-many affine constraints, we can apply the Lagrange multiplier Theorems shown in Section 8.3 and 8.4 of Luenberger (1969) (see Proposition C.4 in the appendix for details). Let λ_i denote the Lagrange multiplier of the intertemporal budget constraint (7) and $\mu_i(\omega)$ the Lagrange multiplier of the incentive compatibility constraint (6). The first-order condition with respect to $c_i(\omega)$ is[15]

$$\pi(\omega)u_i'\left[c_i(\omega)\right] + \mu_i(\omega) = \lambda_i q(\omega). \tag{10}$$

The first term on the left-hand side of (10) reflects that an increase in consumption increases utility and the second term reflects that it relaxes the incentive compatibility constraint (6), while the term on the right-hand side reflects that this increase in consumption tightens the budget constraint.

The first-order condition with respect to N_i^+ is

$$\sum_{\omega\in\Omega} \mu_i(\omega)\delta d_j(\omega) \geq \lambda_i \left[\sum_{\omega\in\Omega} q(\omega)d_j(\omega) - p_j\right], \tag{11}$$

with an equality N_i^+–almost everywhere, that is, for almost all trees held by agent i. The left-hand side of (11) reflects that an increase in i's holdings of tree j increases the amount of non-pledgeable dividend $\delta\, d_j(\omega)$, which tightens the incentive constraint (6) for each ω. The right-hand side reflects that an increase in i's holdings of tree j relaxes the intertemporal budget constraint by $\sum_\omega q(\omega)d_j(\omega) - p_j$.

[15] In principle this condition only holds with an inequality if $c_i(\omega) = 0$, for example when utility is linear. However, we show in the appendix (Proposition C.4) that one can always choose multipliers so that this condition holds at equality.

4.3 Segmentation

Equation (11) can be rewritten

$$p_j \geq \sum_{\omega \in \Omega} q(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \frac{\mu_i(\omega)}{\lambda_i} \delta d_j(\omega) \equiv v_{ij}, \tag{12}$$

with an equality for almost all trees held by agent i, and where v_{ij} can be interpreted as the private valuation of agent i for tree j. The first term of v_{ij}, which is common to all agents, is the present value of the dividends evaluated at state prices. The second term of v_{ij} is agent specific. It reflects the shadow cost incurred by agent i when holding one marginal unit of tree j, thus tightening his incentive constraint. It is the product of the shadow cost of the incentive constraint, $\mu_i(\omega)/\lambda_i$, by the non-pledgeable part of the dividend, $\delta\, d_j(\omega)$, summed across states.

Equation (12) implies that tree j is held by the agents who value it most, because they have the lowest shadow incentive cost of holding this tree. For these agents, $v_{ij} = p_j$, while for the other agents, who do not hold the tree, $v_{ij} \leq p_j$. Thus, differences in private valuations, driven by differences in shadow incentive costs, can lead to market segmentation. This is the case, in particular, when trees have very different payoffs. For example, suppose there is a tree paying off one unit in state ω and zero otherwise and consider some agent i, whose incentive constraint binds in state ω. Then, there exists another agent i' whose incentive constraint is slack in ω, because the incentive constraint cannot bind for all agents in a given state.[16] Therefore $v_{ij} < v_{i'j}$ and agent i does not hold the tree. By continuity, as shown formally in the next proposition, agent i does not hold any tree with large enough dividend in state ω relative to other states.

Proposition 4 *Suppose the equilibrium is not first best, i.e., $\mu_i(\omega) > 0$ for some $i \in I$ and $\omega \in \Omega$. Then, for any agent of type i, there exists $\varepsilon > 0$ and some state ω such that agent i's incentive constraint binds in state ω and agent i strictly prefers not to hold trees j such that $d_j(\omega')/d_j(\omega) < \varepsilon$ for all $\omega' \in \Omega \setminus \omega$.*

The proposition shows that, under natural conditions, segmentation arises in equilibrium. Moreover, it establishes the general principle that an agent will not hold trees making relatively large payments when her incentive constraint binds.

[16] Indeed, adding up the incentive constraint (6) across all agents and using market clearing for consumption, one immediately sees that the aggregate incentive constraint must be slack in each state.

The proposition reveals that the equilibrium distribution of trees across agents ultimately depends on the patterns of binding incentive constraints across states and agents. These patterns are difficult to characterize in full generality because they depend on the distribution of risk aversions across agents, and on the distributions of supplies across trees. But this can be done in the context of particular examples, such as the one developed in Section 5 below.

4.4 Asset Pricing

The pricing of risk and incentives. The first order condition with respect to consumption, (10), shows that if the incentive compatibility conditions were slack, the marginal rate of substitution between consumptions in different states would be equal across all agents, as in the standard, perfect and complete markets, model. When incentive compatibility conditions bind, in contrast, marginal rates of substitution differ across agents, reflecting the multipliers of the incentive constraints. This reflects imperfect risk-sharing in markets that are endogenously incomplete due to incentive constraints, as in Kehoe and Levine (2001). Thus the Arrow securities pricing kernel

$$M(\omega) \equiv \frac{q(\omega)}{\pi(\omega)},$$

which in our model prices the Arrow securities, differs from its perfect and complete markets counterpart because in general, there is no agent whose marginal utility is equal to $M(\omega)$ in all states. Instead, as in Alvarez and Jermann (2000), $M(\omega)$ corresponds to the marginal utility of an unconstrained agent, whose type varies from state to state.

Denote

$$A_i(\omega) \equiv \frac{\mu_i(\omega)}{\lambda_i \pi(\omega)},$$

which can be interpreted as the shadow cost of the incentive compatibility constraint of agent i in state ω. Since the first-order condition (12) holds with an equality for almost all trees held by agent i, we obtain:

$$p_j = \max_{i \in I} v_{ij} = \max_{i \in I} \mathbb{E}\left[M(\omega) d_j(\omega)\right] - \delta\, \mathbb{E}\left[A_i(\omega) d_j(\omega)\right]. \tag{13}$$

Equation (13) shows that the price of a tree is the maximum valuation for the asset across all agents. The price is equal to the present value of the dividends evaluated at Arrow securities prices, minus the shadow incentive-cost of the asset holder.

According to the pricing formula (13), trees with more extreme payoffs, i.e., payoffs close to that of Arrow securities, are less impacted by the shadow incentive-cost of their asset holders and so have relatively higher prices. To see this point, consider a tree in strictly positive supply paying just one unit in state ω and zero otherwise. As argued in Footnote 16, there must exist an agent who is not constrained in state ω. Clearly, equation (12) shows that this agent has the highest valuation for the tree and, correspondingly, equation (13) shows that there is no shadow incentive-cost impounded in the price of the tree. The result that securities with extreme payoffs tend to have higher prices resembles the well-known empirical observation that out-of-the-money calls and puts are expensive (Rubinstein, 1994; Bates, 2000).

Deviations from the Law of One Price. Equation (13) reveals that, if the incentive compatibility constraint of the security holder binds in at least one state, and if the dividend is strictly positive in that state, then the price of the tree is *strictly* smaller than that of the corresponding portfolio of Arrow securities, i.e., there is a basis. One could argue that this constitutes an arbitrage opportunity. However, agents cannot trade on it without tightening their incentive constraint. Thus, the basis between $\mathbb{E}\left[M(\omega)d_j(\omega)\right]$ and the price, p_j, reflects limits to arbitrage induced by incentive constraints.[17]

The basis between trees and replicating portfolios of Arrow securities is a special case of a more general result. Because the maximum operator is convex and v_{ij} is linear in dividends, equation (13) implies that a tree must be priced below *any* replicating portfolio of long positions in trees and/or Arrow securities. The inequality is strict if there is no agent willing to hold all the assets in the replicating portfolio. This is stated formally in the next proposition.

Proposition 5 *For almost all trees j according to $\bar{N}$, consider a replicating portfolio composed of long positions in trees $X \in \mathcal{M}_+$ and Arrow securities $Y \in \mathbb{R}_+^{|\Omega|}$, that is, $d_j(\omega) = \int d_k(\omega)dX_k + Y(\omega)$ for all $\omega \in \Omega$. Suppose there exists no agent willing to hold all the securities in the replicating portfolio, that is,*

[17] Price differences between trees and Arrow securities do not reflect differences in net supply. They arise because trees pay off in different states, while Arrow securities pay only in one state. To see this, consider a tree in strictly positive supply paying one unit in state ω and zero otherwise. There cannot be a basis between the price of that tree and that of the corresponding Arrow security. If there was, then the agent who is not constrained in state ω (there must be one according to Footnote 16) would sell the Arrow security and buy the tree.

there is no $i \in I$ such that $\int v_{ik}\, dX_k = \int p_k\, dX_k$ and $\mu_i(\omega)Y(\omega) = 0$. Then, tree j is priced strictly below its replicating portfolio:

$$p_j < \int p_k\, dX_k + \sum_{\omega \in \Omega} q(\omega)Y(\omega). \tag{14}$$

The economic intuition is that a tree is a bundle of risks that cannot be traded separately from one another, whereas the portfolio of securities with the same payoff as the tree is a bundle of risks that can be traded separately. An agent prefers to be able to choose among the risks in the bundle, retaining only those he wants to bear, and leaving the remaining risks to other agents. This allocation of risks across agents can be interpreted in terms of clientele.

For example, a convertible bond is a combination of a straight bond and a call option on the issuer's stock. In the language of our model, a convertible bond is a tree with the same payoff as a combination of another tree (the straight bond) with a portfolio of Arrow securities (the call option). Our model implies that, if there are no agents who hold simultaneously the straight bond and the call, then the convertible bond should be priced strictly below the price of the straight bond plus the price of the call. In line with our theory, convertible bonds are in fact priced below the replicating portfolio. This deviation from the Law of One Price is at the root of a popular hedge fund strategy ("convertible arbitrage"), which consists in stripping the convertible bond (Mitchell and Pulvino, 2012). Hedge funds buy the convertible bond, issue the set of securities that replicate the convertible bond, and sell the different securities to different clienteles: debt securities are distributed through prime brokers to money market funds and other buyers of safe securities, while equity risk is distributed to equity investors. The convertible arbitrage strategy is constrained, both in practice and in our theory, because arbitrageurs have a limited ability to issue the securities replicating the convertible bond. As a result, convertible bond cheapness increases when arbitrageurs have greater difficulties issuing liabilities, such as during the 1998 LTCM crisis, the 2005 convertible arbitrage meltdown, and the 2008 credit crisis.

A specific implication of our model is that the basis always goes in the same direction. The price of a tree can be lower than that of the replicating portfolio of trees and/or Arrow securities, but it cannot be higher. If it was higher, an agent holding the tree could sell it and buy the replicating portfolio. That arbitrage trade would be feasible because i) market clearing implies there is at least one agent holding the tree, and

ii) replacing a tree by its replicating portfolio does not tighten the IC constraint. In contrast with i), when the price of the tree is lower than that of the replicating portfolio, there does not exist an agent holding the replicating portfolio (since holding that portfolio is dominated). Hence arbitrage trades would require the issuance of liabilities, which would tighten the IC constraint (in contrast with ii)). It is striking that, without any exogenous difference in margin constraints or pledgeability between trees and their replicating portfolios, the former are priced below the latter.

Excess return decomposition. The pricing formula (13) leads to a natural decomposition of excess return. Define the risky return on security j as $R_j(\omega) \equiv d_j(\omega)/p_j$ and let the risk-free return be $R_f \equiv 1/\mathbb{E}[M(\omega)]$. Then, standard manipulations of the first order condition (12) show that for almost all securities held by agents of type i:

$$\mathbb{E}[R_j(\omega)] - R_f = -R_f \operatorname{cov}[M(\omega), R_j(\omega)] + R_f \, \mathbb{E}[A_i(\omega)\delta R_j(\omega)] \tag{15}$$

The first term on the right-hand side of (15) can be interpreted as a risk premium. It is positive if the return on tree j, $R_j(\omega)$, is large for states in which the pricing kernel, $M(\omega)$, is low. It is similar to the standard risk premium associated with consumption betas in frictionless markets but, unlike in the frictionless CCAPM, the pricing kernel $M(\omega)$ mirrors neither aggregate nor individual consumption.

The second term on the right-hand side of (15) is a premium reflecting incentive constraints. This premium is large if non-pledgeable income, $\delta R_j(\omega)$, is large when the incentive compatibility condition is tight.

While equation (15) bears some similarities with the margin-CAPM characterized by Gârleanu and Pedersen (2011), it differs from it in two important ways. First, as shown by the first term of equation (15), the consumption beta in our model is defined relative to the consumption of an unconstrained agent, whose identity differs across states. Second, as shown by the second term of equation (15), the premium associated with financial constraints differs across securities, even though all securities have the same margin requirement, δ. This is because the financial constraints in our model are endogenously state contingent. Correspondingly the premium depends on the covariance between the state-contingent non-pledgeable income generated by the security, and the state-contingent shadow price of the constraint.

Basis vs. collateral premium. Equation (13) shows that the tree is priced at a discount relative to the replicating portfolio of Arrow securities, i.e., there is a basis. This could seem to contradict the result obtained in previous literature (see, e.g., Fostel and Geanakoplos, 2008) that equilibrium prices include a collateral premium. There is no contradiction, however, as the premium obtains, both in this paper and in previous literature, relative to a different benchmark than the replicating portfolio.

To see this, consider again the trees held by some agent i. Take the first-order condition (10) with respect to $c_i(\omega)$, multiply by the dividend $d_j(\omega)$ and sum across states to obtain:

$$\mathbb{E}\left[M(\omega)d_j(\omega)\right] = \mathbb{E}\left[\frac{u_i'\left[c_i(\omega)\right]}{\lambda_i}d_j(\omega)\right] + \mathbb{E}\left[A_i(\omega)d_j(\omega)\right]. \tag{16}$$

Substituting (16) into (13) the price of tree j is

$$p_j = \mathbb{E}\left[\frac{u_i'\left[c_i(\omega)\right]}{\lambda_i}d_j(\omega)\right] + \Big(\mathbb{E}\left[A_i(\omega)d_j(\omega)\right] - \delta\,\mathbb{E}\left[A_i(\omega)d_j(\omega)\right]\Big). \tag{17}$$

This price equation is similar to equation (5) in Fostel and Geanakoplos (2008) or that in Lemma 5.1 in Alvarez and Jermann (2000). The first term on the right-hand side of (17) is similar to what Fostel and Geanakoplos (2008) call "payoff value": it is the expected value of the tree cash flows, evaluated at the marginal utility of the agent holding it. The second term on the right-hand side of (17) is similar to the collateral premium in Fostel and Geanakoplos (2008) (see Lemma 1, page 1230). This premium, however, is reduced by the last term (in factor of δ) which corresponds to the basis whose analysis is one of the contributions of our paper.

5 Two-by-Two

In the previous section, we derived some key implications of our model but did not characterize the equilibrium completely. In particular, we did not derive the endogenous patterns of binding incentive constraints across states and agents, which ultimately determine the equilibrium distribution of trees. In this section, we provide a full characterization of equilibrium in the natural "two-by-two" case. Namely, we assume that there are two types of agents $i \in \{1,2\}$, two states, $\omega \in \{\omega_1, \omega_2\}$, and an arbitrary distribution of trees. We further assume that both types of agents, $i \in \{1,2\}$, have CRRA utility with respective coefficients of relative risk aversion $0 \le \gamma_1 < \gamma_2 \le 1$. That is, agent $i = 1$ is more risk tolerant, while agent $i = 2$ is more risk averse.[18]

We normalize the dividend of each tree to one, i.e., $\mathbb{E}[d_j(\omega)] = 1$.[19] Given that there are only two states, any tree can be viewed as a convex combination of two extreme securities: one security that only pays off in state ω_1, and one security that only pays off in state ω_2. Therefore, one can order the trees so that, for any $j \in [0,1]$,

$$d_j(\omega) = \frac{j}{\pi(\omega_1)} \mathbb{I}_{\{\omega=\omega_1\}} + \frac{1-j}{\pi(\omega_2)} \mathbb{I}_{\{\omega=\omega_2\}}. \tag{18}$$

Notice that, after the normalization $\mathbb{E}[d_j(\omega)] = 1$, in a two-state model the payoff of any security can be represented as (18) for some j.

We label the states so that the aggregate endowment, denoted by $y(\omega) = \int d_j(\omega)\, d\bar{N}_j$, is strictly larger in state ω_2 than in state ω_1:

$$y(\omega_2) = \frac{1}{\pi(\omega_2)} \int (1-j)\, d\bar{N}_j > y(\omega_1) = \frac{1}{\pi(\omega_1)} \int j\, d\bar{N}_j. \tag{19}$$

In other words, ω_1 is the "bad state" while ω_2 is the "good state." The tree $j = \pi(\omega_1)$ is risk free, and so its consumption beta, $\mathrm{cov}[d_j(\omega), y(\omega)] / \mathrm{var}[y(\omega)]$ is zero. Trees with $j < \pi(\omega_1)$ have lower dividend in state ω_1 than in state ω_2, and so have positive consumption beta. The smaller is j, the more positive is the beta. Vice versa, trees with $j > \pi(\omega_1)$ have negative consumption beta. The larger is j, the more negative is the beta. Hence, the parameter j can be interpreted as an index of safety.

[18] As shown in Proposition C.8 in the appendix, $0 \le \gamma_1 \le \gamma_2 \le 1$ and $\gamma_2 > 0$ imply that the equilibrium consumption allocation is uniquely determined, and the equilibrium prices are uniquely determined up to a multiplicative constant. As shown in Proposition C.7 in appendix, the restriction $\gamma_1 \ne \gamma_2$ is necessary for incentive compatibility to bind in equilibrium.

[19] This is without loss of generality. This merely amounts to divide the dividend in all states by the expected dividend, and simultaneously scaling the tree supply up by the same constant.

5.1 Incentive Feasible Consumption Allocations

In order to characterize the patterns of binding incentive constraints across states and agents, we take a step back and study the set of *incentive feasible* consumption allocations, that is, consumption allocations $(c_i)_{i\in\{1,2\}}$ such that $(c_i, N_i^+)_{i\in\{1,2\}}$ is incentive compatible for some tree allocation $(N_i^+)_{i\in\{1,2\}}$. As will become clear, this turns out to be very useful: indeed, it allows to characterize incentive feasibility as a function of the consumption allocation only, and leave the corresponding feasible allocation of trees implicit.

Focusing on the case in which the consumption share of agent 1 is lower in the bad state than in the good state (which, as shown below, is the case in equilibrium), our first main result is the following:

Lemma 2 *Consider a feasible consumption allocation such that $c_1(\omega_1)/y(\omega_1) < c_1(\omega_2)/y(\omega_2)$. Then, c is incentive feasible if and only if there exists $k \in [0,1]$ and $(\Delta N_1, \Delta N_2) \geq 0$, $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$, such that:*

$$
\begin{aligned}
c_1(\omega_1) \quad &\geq \quad \delta \int_{j\in[0,k)} d_j(\omega_1) d\bar{N}_j + \delta d_k(\omega_1)\Delta N_1 &\quad (20)\\
c_2(\omega_2) \quad &\geq \quad \delta \int_{j\in(k,1]} d_j(\omega_2) d\bar{N}_j + \delta d_k(\omega_2)\Delta N_2. &\quad (21)
\end{aligned}
$$

Equation (20) is the incentive compatibility condition of agent $i = 1$ in state ω_1 when he holds all trees riskier than k ($j < k$), and, if there is an atom at k in the distribution of trees, a mass ΔN_1 of that atom. Similarly, equation (21) is the incentive compatibility condition of agent $i = 2$ when he holds all trees $j > k$, plus a mass ΔN_2 of the atom at k, if there is one. The lemma thus states that a consumption allocation such that $c_1(\omega_1)/y(\omega_1) < c_1(\omega_2)/y(\omega_2)$ is incentive feasible if and only if it can be implemented by allocating riskier trees to the more risk tolerant agent 1 and safer trees to the more risk averse agent 2 without violating their incentive compatibility constraints in states ω_1 and ω_2, respectively. The intuition for this result can be grasped from the following two observations.

The first observation is that, since her consumption share is smaller in ω_1 than in ω_2, agent $i = 1$ tends to have incentive problems in state ω_1. To understand why, imagine that agent $i = 1$ purchases a fraction of the market portfolio equal to her average consumption share across states. In order to implement her consumption plan $c_1(\omega)$ while holding this portfolio, agent $i = 1$ has to sell Arrow securities that pay off in state ω_1, and purchase Arrow securities that pay off in state ω_2. Hence, agent $i = 1$ only has a liability in state ω_1, so that her incentive compatibility constraint can bind only in that state. Vice versa, agent $i = 2$ faces incentives problems only in state ω_2. The lemma states that only two (equations (20) and (21)) out of the four incentive compatibility constraints matter.

The second observation is that, to mitigate these incentive problems, it is best to allocate agent $i = 1$ a portfolio of trees with low payoff in state ω_1. This minimizes the amount this agent can renegotiate in the state in which his incentive constraint binds. Symmetrically, it is best to allocate agent $i = 2$ a portfolio of trees with low payoff in state ω_2. By market clearing an equivalent way to grasp the intuition for this result is the following: Allocating to the more risk-averse agents trees with relatively high payoff in the bad state reduces the reliance of that agent on insurance sold by the more risk tolerant agent. This, in turn, relaxes incentives constraints.

The lemma is illustrated in the Edgeworth box in Figure 1. The consumption of agent $i = 1$ in state ω_1 is on the x-axis, and his consumption in state ω_2 is on the y-axis. The curves above and below the diagonal are the boundaries of the incentive feasible set. In line with the lemma, focus on the area above the diagonal, where the consumption share of agent 1 is larger in the good state, ω_2, than in the bad state ω_1. All the area between the boundary of the incentive set and the diagonal is incentive compatible.

One sees in the figure that any allocation which gives sufficiently small consumption to one of the agents is incentive feasible. For example, if the consumption of agent $i = 1$ is sufficiently small then the consumption of agent $i = 2$ is almost equal to the aggregate endowment. As long as $\delta < 1$, this allocation can be made incentive feasible by allocating all the trees to agent $i = 2$. In equilibrium, agent $i = 1$ sells all his trees to agent $i = 2$, and agent $i = 2$ issues a liability corresponding to agent $i = 1$ consumption. This is feasible since $\delta < 1$ gives agent $i = 2$ some borrowing capacity.

Figure 1 also compares the incentive sets for one tree and for many trees (keeping aggregate output in each state constant.) The dashed line is the boundary of the incentive-feasible set when there is just one tree in strictly positive supply.[20] The solid line is the boundary when there are many trees.[21] The figure

[20] In that case, the distribution $\bar{N}$ has just one atom. If we normalize this atom to one for simplicity, then in the Edgeworth box the boundary is the curve parameterized by $\Delta N_1 \in [0, 1]$, with cartesian coordinates $c_1(\omega_1) = \delta d(\omega_1)\Delta N_1$ and $c_1(\omega_2) = y(\omega_2) - c_2(\omega_2) = d(\omega_2)\,[1 - \delta + \delta \Delta N_1]$

[21] In that case we assume no atom, so the boundary is the curve parameterized by $k \in [0, 1]$, with cartesian coordinates $c_1(\omega_1) = \delta \int_0^k d_j(\omega_1)\, d\bar{N}_j$ and $c_1(\omega_2) = y(\omega_2) - c_2(\omega_2) = \int_0^1 d_j(\omega_2)\, d\bar{N}_j - \delta \int_k^1 d_j(\omega_2) d\bar{N}_j$.

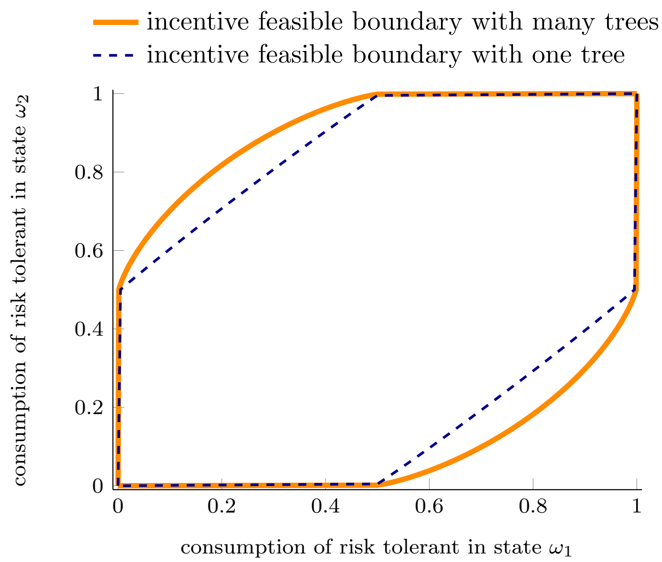

Figure 1: The set of incentive feasible consumption allocations in an Edgeworth box, when $\pi(\omega_1) = 0.1$ and $\delta = 0.5$. In the many-trees case, tree supplies are distributed according to a beta distribution with parameters $a = b = 5$. In the one-tree case, there is just one tree equal to the market portfolio of the many-trees case.

illustrates that the incentive-feasible set is smaller with one tree than with many trees. Indeed, with many trees, one can replicate one-tree allocations by allocating agents shares in the market portfolio.

5.2 Equilibrium Allocations

In order to characterize equilibrium allocations, we rely on their efficiency properties. Let $(c_i, N_i^+)_{i\in\{1,2\}}$ denote the equilibrium allocation. As shown in Proposition 2, $(c_i, N_i^+)_{i\in\{1,2\}}$ is constrained Pareto efficient. That is, $(c_i, N_i^+)_{i\in\{1,2\}}$ solves an *incentive-constrained* planner's problem, i.e., there exist weights $(\alpha_1, \alpha_2) \in (0,1)^2$, $\alpha_1 + \alpha_2 = 1$, such that c maximizes $\sum_{i\in I} \alpha_i U_i(c_i)$ with respect to feasible allocations satisfying the incentive compatibility conditions. Let $c^\star$ denote the solution of the corresponding *unconstrained* planner's problem. That is, $c^\star$ maximizes the same welfare function, with the same weights (α_1, α_2), with respect to feasible allocations, but without imposing the incentive compatibility conditions.

Lemma 3 *If $(\alpha_1, \alpha_2) > 0$, then the solutions of the unconstrained and incentive-constrained planner's problems both are such that $c_1^\star(\omega_1)/y(\omega_1) < c_1^\star(\omega_2)/y(\omega_2)$ and $c_1(\omega_1)/y(\omega_1) < c_1(\omega_2)/y(\omega_2)$.*

The lemma states that the more risk tolerant agent, $i = 1$, receives a lower share of aggregate consumption in the low state than in the high state (as in the first best). Since consumption shares add up to one across agents, it follows that the risk averse agent, $i = 2$, enjoys a higher share of aggregate consumption in the low than in the high state. Intuitively, a consumption allocation which delivers a constant consumption share in both states to both agents is always strictly incentive feasible since it lies on the diagonal of the Edgeworth box of Figure 1. But the risk tolerant cares relatively less about the low state, ω_1, and relatively more about the high state, ω_2. Hence, social welfare increases strictly if the risk tolerant agent, $i = 1$ insures the more risk averse agent by letting $i = 2$ have a larger share of aggregate consumption in the bad state.

Lemma 3 states that the planner always finds it optimal to pick consumption allocations above the diagonal of the Edgeworth box. Therefore, the relevant incentive constraint is the upper boundary of the incentive feasible set in Figure 1. Using Lemma 2, we then obtain our next proposition:

Proposition 6 *If $c \neq c^\star$, then the incentive compatibility constraint of agent $i = 1$ binds in state ω_1, while the incentive compatibility constraint of agent $i = 2$ binds in state ω_2. Moreover, there exists $k \in [0, 1]$ and $(\Delta N_1, \Delta N_2) \geq 0$, $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$, such that agent $i = 1$ holds all trees $j < k$, i.e.,*

$$N_1^+ = \bar{N}\, \mathbb{I}_{\{j<k\}} + \Delta N_1 \mathbb{I}_{\{j=k\}}$$

and agent $i = 2$ holds all trees $j > k$, i.e., $N_2^+ = \bar{N}\, \mathbb{I}_{\{j>k\}} + \Delta N_2 \mathbb{I}_{\{j=k\}}$.

Lemma 2 stated that a consumption allocation was incentive feasible if and only if it could be implemented by allocating the riskier trees to the more risk tolerant agent and the safer trees to the more risk averse agent. In line with that result, Proposition 6 states that, in equilibrium, the binding incentive constraints of the more risk tolerant agent in the bad state, and the more risk averse agent in the good state, pin down such an allocation. This equilibrium allocation can be interpreted in terms of segmentation, as different classes of investors hold different types of trees.

The proposition is illustrated in Figure 2. In the figure, the "constrained Pareto set" and the "unconstrained Pareto set" are, respectively, the set of consumption allocations obtained by solving the incentive-constrained and the unconstrained Planner's problem for all possible weights $(\alpha_1, \alpha_2) \in [0, 1]^2$, $\alpha_1 + \alpha_2 = 1$. The incentive-constrained Pareto set coincides with the unconstrained Pareto set when the latter lies below the upper boundary of the incentive-feasible set. Otherwise, the incentive-constrained Pareto set coincides

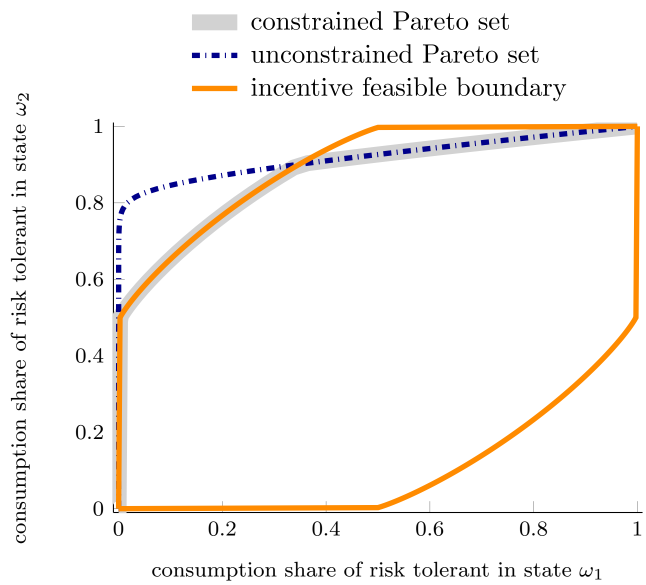

Figure 2: The set of incentive-feasible and the set of incentive-constrained Pareto allocation, when $\pi(\omega_1) = 0.9$, $\gamma_1 = 0.1$, $\gamma_2 = 1$, and $\delta = 0.5$. In the many-trees case, tree supplies are distributed according to a beta distribution with parameters $a = b = 15$. In the one-tree case, there is just one tree equal to the market portfolio of the many-trees case.

with the IC boundary. As α_1/α_2 increases, then the constrained Pareto efficient allocation moves monotonically to the northeast of the Edgeworth box of Figure 1.

Next, we translate the above discussion into equilibrium comparative statics. To do so, all we need is to study the mapping between the exogenous endowments $(\bar{n}_1, \bar{n}_2)$ and their corresponding *endogenous Pareto weights*, that is, the weights (α_1, α_2) such that the equilibrium allocation given endowments $(\bar{n}_1, \bar{n}_2)$ is the solution of the incentive-constrained Planner's problem. The existence of endogenous Pareto weights is immediate from Propositions 2 and 3. We obtain:

Proposition 7 *The ratio of endogenous Pareto weights, α_1/α_2, is strictly increasing in the ratio of initial endowment $\bar{n}_1/\bar{n}_2$.*

Thus, while Figure 2 reveals that incentive compatibility does not matter for extreme values of α_1/α_2, Proposition 7 enables one to restate this observation in terms of the distribution of initial endowments. When this distribution is very unequal, as agents $i = 1$ are initially endowed with a very large fraction of the market portfolio ($\bar{n}_1/\bar{n}_2$ large), or agents $i = 2$ are initially endowed with a very large fraction of the market portfolio ($\bar{n}_1/\bar{n}_2$ low), there is little scope for risk sharing between the two types of agents. Thus, even the unconstrained equilibrium involves little trading, so that incentive constraints do not bind (as is the case in Figure 2 in the north east and the south west of the Edgeworth box). In contrast, when the distribution of initial endowments is more equal ($\bar{n}_1$ close to $\bar{n}_2$) the scope for risk sharing is large. In that case the incentive-constrained equilibrium allocation differs significantly from its unconstrained counterpart (as is the case in Figure 2 in the north west of the Edgeworth box). Correspondingly, the basis is zero when the distribution of endowments is very unequal, while it can be strictly positive when initial endowments are equally distributed.

5.3 Relative Supply Effects

In our model, the relative supply of trees (i.e., the distribution $\bar{N}_j$) determines equilibrium outcomes, by changing the shape of the incentive feasible set. Changing the relative supplies of trees changes equilibrium outcomes even if it does not change aggregate output and pledgeable income in each state, nor the span of the trees' payoff matrix. This is in sharp contrast with the perfect and complete markets case, in which the set of feasible allocations is not affected by the way the aggregate output is split across trees.

As an illustration, compare an economy with just one tree (the "market portfolio") to another economy in which there are two trees, one paying off only in the good state, the other paying off only in the bad state. To reason ceteris paribus, the aggregate output is the same in each state in the two economies. If incentive problems are severe, because $\delta \simeq 1$, then, when there is only one tree, incentive constraints are more likely to bind in equilibrium. In contrast, when there are two trees, each paying off only in one state, all agents can attain their $\delta = 0$ equilibrium consumption just by holding trees. Hence incentive constraints do not bind, even if $\delta \simeq 1$.[22]

When there is a single tree, its consumption β is equal to one, while with two trees one has a lower β and the other a higher one. The comparison between the single tree economy and its two tree counterpart suggests that increasing the dispersion of βs relaxes incentive constraints. To make this point, first note that in our simple two-state case, the consumption β of tree j is affine in j with coefficients that only depend on the values and probabilities of aggregate dividends. Therefore, the dispersion of βs is increasing in the dispersion of js.

To model an increase in the dispersion of the distribution of trees, from $\bar{N}$ to $\bar{N}^*$, it is natural to consider

[22] A proof of this and related results is provided in the appendix in Proposition C.7.

a mean preserving spread, that is, a decrease in the sense of second order stochastic dominance:

$$\int_0^k \bar{N}_j^* \geq \int_0^k \bar{N}_j, \qquad \text{for all } k \in [0,1], \tag{22}$$

preserving the aggregate output in each state. From equation (19), one sees that aggregate output is preserved in each state if and only if:

$$\int_0^1 j \, d\bar{N}_j^* \quad = \quad \int_0^1 j \, d\bar{N}_j \tag{23}$$

$$\bar{N}_1^* \quad = \quad \bar{N}_1. \tag{24}$$

In this context, we obtain the following proposition:

Proposition 8 *Consider two tree supply distributions $\bar{N}$ and $\bar{N}^*$, such that $\bar{N}^*$ is more dispersed than $\bar{N}$ in the sense that (22), (23) and (24) hold. If an allocation is incentive feasible for $\bar{N}$, then it is incentive feasible for $\bar{N}^*$.*

Proposition 8 states that the set of incentive feasible allocations expands as the tree supply distribution becomes more dispersed, i.e., the dispersion of βs increases. The proposition enables one to revisit and generalise the intuition obtained by comparing the single tree and two-tree economies: a mean-preserving spread implies that the supply distribution $\bar{N}^*$ puts more weight on the riskiest trees, and also on the safest trees, than the distribution $\bar{N}$. This implies that the set of trees held by the more risk tolerant agent is overall riskier, and correspondingly yields smaller dividends in the bad state, which relaxes the incentive compatibility condition of this agent. Symmetrically, by putting more weight on the safest trees, the supply distribution $\bar{N}^*$ leads to an overall safer set of trees held by the more risk averse agent. This reduces the dividends of these trees in the good state, which relaxes the incentive constraint of the more risk averse agent.

The tightening of incentive constraints, induced by a decrease in the dispersion of consumption betas, affects equilibrium pricing. This is illustrated in Figure 3, which plots the equilibrium expected return on the market portfolio for different dispersions of consumption betas, keeping aggregate output in each

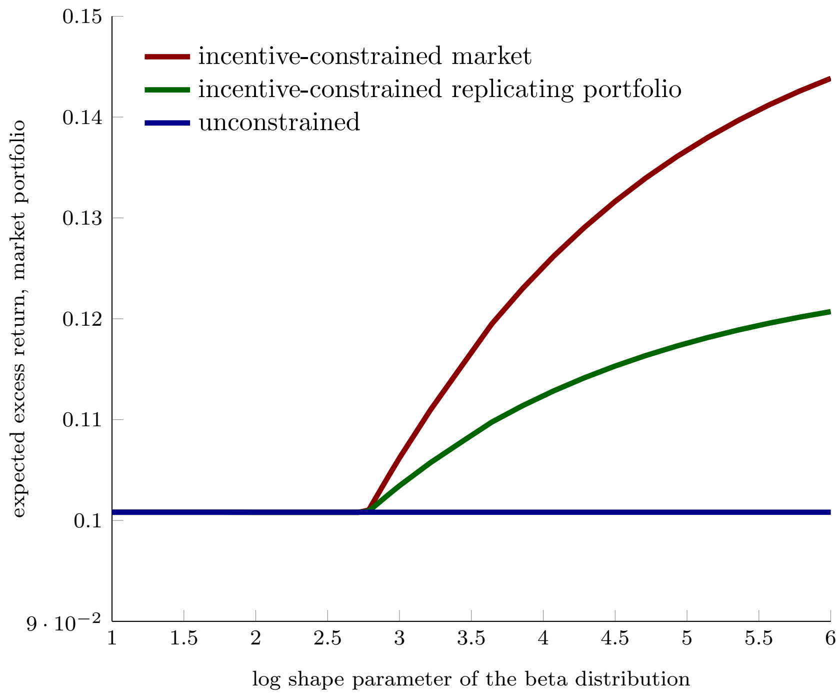

Figure 3: The expected excess return on the market portfolio as the dispersion of security beta decreases. Tree supplies are distributed according to a beta distribution with shape parameters $a = b$ varying, in logs, from 1 to 6.

state constant. We consider symmetric beta distributions, and let the log of their shape parameter increase from 1 to 6, so that their dispersion decreases while their mean remains constant. The figure shows that, as the dispersion of consumption betas decreases, the expected return of the portfolio of Arrow securities replicating the market increases, reflecting poorer allocation of risks in the economy. Moreover, as betas get less dispersed, the basis increases, reflecting the increased shadow costs of incentive constraints. In contrast, by construction, when there are no incentive problems the expected return on the market portfolio is not affected by changes in the dispersion of betas.

5.4 The Cross Section of Bases

Focus on the case in which incentive compatibility constraints bind. By Proposition 6, there exists a threshold k, such that the more risk tolerant agent holds trees $j < k$, while the more risk averse agent holds trees $j > k$. In this context, from the first-order condition (12), the basis on tree $j < k$ held by the risk tolerant agent, $i = 1$, is

$$\sum_{\omega \in \Omega} q(\omega) d_j(\omega) - p_j = \delta \frac{\mu_1(\omega_1)}{\lambda_1} d_j(\omega_1).$$

Since only relative prices are pinned down, we express this basis in relative prices, and choose as normalizing factor (or numeraire) the price of the riskless bond, $q(\omega_1) + q(\omega_2)$. For tree $j < k$ the normalised basis is thus

$$\Delta_j = \Big[\frac{\mu_1(\omega_1)/\lambda_1}{q(\omega_1) + q(\omega_2)}\Big] \times \Big[\delta\, d_j(\omega_1)\Big]. \tag{25}$$

The first term in the right-hand side of equation (25) is constant across all trees held by agent 1, and measures, intuitively, the tightness of the incentive constraint of agent 1. The second term is equal to the non-pledgeable cash flow of the tree in state ω_1 in which the agent holding it is constrained. This term, and correspondingly the basis, is higher for trees with a relatively large payoff in the bad state and a relatively low payoff in the high state, that is, trees with a lower consumption beta. The intuition is that the risk tolerant agent sells insurance against the bad state to the risk averse agent. However, the incentive compatibility constraint limits the amount of insurance she can sell. Since the consumption of the risk tolerant agent is low in the bad state, renegotiating is tempting. It implies that the shadow cost of holding a tree is higher for trees paying relatively more in the bad state, i.e., for trees with a lower consumption beta. Remember however that the risk tolerant agent holds trees with a high beta. Therefore, among trees with a high consumption beta, trees with a moderately high beta have a larger basis than trees with a very high beta.

Consider now trees $j > k$ held by agent 2. Following the same reasoning as before, the basis equals

$$\Delta_j = \Big[\frac{\mu_2(\omega_2)/\lambda_2}{q(\omega_1) + q(\omega_2)}\Big] \times \Big[\delta\, d_j(\omega_2)\Big]. \tag{26}$$

Equation (26) implies that, among trees held by the risk averse agent, $i = 2$, the basis is larger for trees with a relatively large payoff in the good state and a relatively low payoff in the bad state, that is, with a higher consumption beta. The intuition is symmetric to the one above. The risk averse agent would like to sell consumption to the risk tolerant agent in the good state, but it is tempting for the risk averse agent to renegotiate in the good state. Thus, the shadow cost of holding a tree is higher for tree with a relatively high payoff in the good state, that is, for trees with a higher consumption beta. The risk averse agent holds

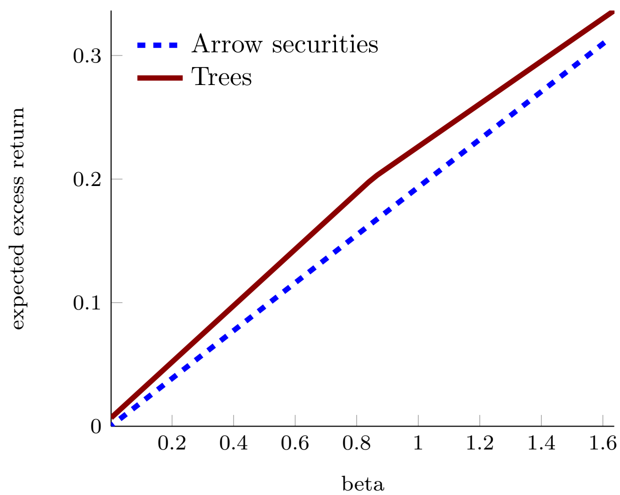

Figure 4: The Modified Security Market Line

trees with a low consumption beta. Therefore, among trees with a low beta, those with a moderately low beta have a larger basis than trees with a very low beta. Putting things together, we conclude that:

Proposition 9 *Suppose all trees are in strictly positive supply. Then, the basis induced by incentive constraints is an inverse U-shape function of the consumption beta of the tree.*

The restriction that all trees are in positive supply ensures that prices are uniquely determined. Intuitively, the proposition means that, after adjusting for risk, trees with either a low or a large consumption beta will tend to have a high price, and a low return. This is illustrated in Figure 4. The figure shows the security market line (SML) in our environment, which we derive explicitly in Proposition E.1 in the appendix. Since trees are held by agents who value them most, the SML is the minimum between the SML obtained from agent $i = 1$'s valuation, and that derived from agent $i = 2$'s valuation. The kink in the figure occurs at tree k, for which ownership switches from agent 1 to agent 2. The figure illustrates that, because the basis is inverse-U shaped in β, the SML is flatter at the top and steeper at the bottom, in line with Black (1972), and recent evidence in Frazzini and Pedersen (2014) and Hong and Sraer (2016).

Finally, we re-cast our findings in an option-pricing context. Recall that, following a standard binomial option-pricing argument, a bond is replicated by a portfolio made up of a stock and of a put option. Now consider a sufficiently risky stock (low j or high beta) and an out-of-the-money put, interpreted as an Arrow security with positive payoff in the low state, and zero payoff in the high state. Then the stock is held by the risk tolerant agent while the put option, which only pays off in the low state, must be held by the risk averse agent. It thus follows from Proposition 5 that the bond is priced strictly below the replicating portfolio: that is, in line with empirical evidence, out-of-the-money puts appear too expensive relative to standard arbitrage relationships. A symmetric reasoning reveals that, if risk-free bonds are held by risk averse agents, then out-of-the-money calls also appear too expensive. According to our model, these price discrepancies arise because of limits to arbitrage induced by incentive constraints. For example, to take advantage of the price discrepancy, the risk tolerant agent would need to sell the high beta stock, and sell more of the put, which he cannot do because of the incentive constraint.

5.5 Response to Shocks on Incentive Problems

To model shocks on incentives, fix a tree $\ell < k$ and consider a small increase in δ for tree ℓ and possibly nearby trees. Formally, assume $\delta_j = \delta + \varepsilon\phi_j$ for some continuous function ϕ_j strictly positive near ℓ, and zero everywhere else.[23] The shock worsens incentive problems for tree ℓ (and neighbouring trees). What is the effect of this shock on allocations?

Proposition 10 *Assume that the cumulative distribution of trees is continuous and strictly increasing, that $c \neq c^\star$, and that $k \in (0, 1)$. The ε shock shrinks the set of trees held by agent 1: $k(\varepsilon) < k(0)$ for small $\varepsilon > 0$.*

When agent 1 becomes slightly worse at pledging a tree he already holds, the shadow value of his incentive-compatibility constraint increases, which makes it more costly for the more risk tolerant agent 1 to hold any tree. Thus, in equilibrium, the set of trees $[0, k)$ held by agent 1 shrinks. This means that agent $i = 1$ sells the safest trees in his portfolio, while keeping the riskiest ones. This might sound counter-intuitive if one expected that, when an agent's incentive problems become more severe, he should sell his riskiest trees. The result arises because, even after the shock, agent $i = 1$ is still in a better position to hold very risky trees than the more risk averse agent $i = 2$. Thus, as the shock reduces agent $i = 1$'s ability to hold trees overall, he divests those for which his comparative advantage is the lowest.

Now turn to the effect of the shock on prices. Equations (25) and (26) imply that when agent $i = 1$ becomes a worse pledger for tree ℓ, the basis of tree ℓ increases relative to the bases of other trees j held by

[23] All of our results extend to this case. In fact, our proofs in the appendix cover the case of δ which are continuously varying across agents and tree types.

$i = 1$ (that have not been directly hit by the shock), which themselves increase relative to the bases of trees j' held by $i = 2$. That is

$$\frac{\Delta_\ell(\varepsilon)}{\Delta_\ell(0)} > \frac{\Delta_j(\varepsilon)}{\Delta_j(0)} > \frac{\Delta_{j'}(\varepsilon)}{\Delta_{j'}(0)}$$

Thus, co-movement in the basis induced by incentive constraints is stronger among trees held by the same type of agents.

6 Conclusion

We introduce incentive compatibility constraints, limiting the pledgeability of collateral, in an otherwise standard general equilibrium model. In each state, agents cannot pledge more than a fraction of the payoff from their holdings in that state. Hence, although a complete set of Arrow securities are available for trade, limited collateral pledgeablity drives risk-sharing below its first best counterpart. Thus, markets are endogenously incomplete.

To cope with incentive constraints, relatively risk averse agents hold low beta securities, while relatively risk tolerant agents hold high beta securities. This reflects the risk tolerant agents' comparative advantage at holding risky securities. Correspondingly, the market is endogenously segmented.

When hit by an adverse shock on incentives, relatively risk tolerant agents sell their safest securities, not their riskiest ones. This is because, while the shock reduces their ability to hold all securities, it does not eliminate their comparative advantage at holding the riskiest ones.

Incentive compatibility constraints generate a basis between the prices of risky securities and those of replicating portfolios of derivatives (in spite of the fact that risky securities and derivatives are equally imperfectly pledgeable). The basis always goes in the same direction: the price of risky securities is below that of replicating derivative portfolios. Arbitraging the basis would imply buying the "cheap" risky securities, and selling the "expensive" derivatives, but the latter sale is ruled out by incentive compatibility. The structure of the basis is such that equilibrium expected excess returns are concave in consumption betas, in line with empirical findings. Moreover an increase in the dispersion of consumption betas relaxes incentive constraints, which reduces the basis. The latter two results are related. The concavity of the security market line reflects that securities with extreme betas are more valuable because they enable agents to share risk without creating incentive problems. An increase in the dispersion of consumption betas increases the prevalence of securities with extreme betas, which relaxes incentive constraints in equilibrium.

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