Incentive Constrained Risk Sharing, Segmentation, and Asset Pricing
Online Appendix

Bruno Biais, Johan Hombert, and Pierre-Olivier Weill

I Proof of Lemma A.1

It is clear that $\overline{\mathbb{NA}}$ is included in the set defined by (35), (36), (37) with weak inequalities. For the reverse inclusion, consider any (p, q) in the set defined by (35), (36), (37) with weak inequalities, and any $\varepsilon > 0$. Let

$$q_t^{\varepsilon}(s^t) = q_t(s^t) + \varepsilon,$$

and, working backward from time $T-1$:

$$p_t^{\varepsilon}(\delta \; s^t) = p_t(\delta \,|\, s^t) + \varepsilon \sum_s \delta_{t+1}(s^t, s) + \sum_s \left(p_{t+1}^{\varepsilon}(s^t, s) - p_{t+1}(s^t, s) \right), \tag{54}$$

with the convention that $p_T(s^T) = p_T^{\varepsilon}(s^T) = 0$. We prove by backward induction that $(p^{\varepsilon}, q^{\varepsilon})$ satisfies the no-arbitrage conditions (35)-(36)-(36), as well as:

$$p_t^{\varepsilon}(\delta \,|\, s^t) \geq p_t(\delta \,|\, s^t) \text{ with } > \text{ if } \delta \in \Delta_t^+(s^t). \tag{55}$$

First, notice that, (35) holds by construction at all (t, s^t), while (35), (36), (37) and (55) hold by construction at time T. Now we show that they must hold at time t if they hold at time $t+1 \leq T$.

- Evidently, since (55) holds at $t+1$, the definition of $p_t^{\varepsilon}(s^t)$ implies that holds with weak inequality at time t for all δ. It clearly holds with a strict inequality for all $\delta \in \Delta_t^+(s^t)$ such that $\delta_{t+1}(s^t, s) > 0$ for some s. If $\delta \in \Delta_t^+(s^t)$ but $\delta_{t+1}(s^t, s) = 0$ for all s, then $\delta \in \Delta_{t+1}^+(s^t, s)$ for some s, and the strict inequality follows from (54) together with our induction hypothesis that (55) holds at time $t+1$.
- Since (36) holds with a weak inequality for (p, q):

$$\begin{aligned} p_t^{\varepsilon}(\delta \,|\, s^t) &\leq \sum_s \left[q_{t+1}(s^t, s)\delta_{t+1}(s^t, s) + p_{t+1}(\delta \,|\, s^t, s) \right] + \varepsilon \sum_s \delta_{t+1}(s^t, s) \\ &\quad + \sum_s \left[p_{t+1}^{\varepsilon}(\delta \,|\, s^t, s) - p_{t+1}(\delta \,|\, s^t, s) \right] \\ &= \sum_s \left[q_{t+1}^{\varepsilon}(s^t, s)\delta_{t+1}(s^t, s) + p_{t+1}^{\varepsilon}(\delta \,|\, s^t, s) \right]. \end{aligned}$$

  Hence (36) holds for $(p^{\varepsilon}, q^{\varepsilon})$ at time t.

- Likewise, since (37) holds with weak inequality for (p, q):

$$\begin{aligned}
p_t^{\varepsilon}(\delta \mid s^t) \geq &(1-\theta_i(\delta)) \sum_s \left[q_{t+1}(s^t, s)\delta_{t+1}(s^t, s) + p_{t+1}(\delta \mid s^t, s)\right] + \varepsilon \sum_s \delta_{t+1}(s^t, s) \\
&+ \sum_s \left[p_{t+1}^{\varepsilon}(\delta \mid s^t, s) - p_{t+1}(\delta \mid s^t, s)\right] \\
= &(1-\theta_i(\delta)) \sum_s \left[q_{t+1}^{\varepsilon}(s^t, s)\delta_{t+1}(s^t, s) + p_{t+1}^{\varepsilon}(\delta \mid s^t, s)\right] \\
&+ \theta_i(\delta) \left[p_t^{\varepsilon}(\delta \mid s^t) - p_t(\delta \mid s^t)\right].
\end{aligned}$$

Together with the fact, shown in the first bullet point, that (55) holds at time (t, s^t), we obtain (37).

After scaling down q^{ε} and p^{ε} by the same constant so that the normalization (34) holds, we obtain a price system in $\mathbb{NA}$. This price system clearly converges to (p, q) as $\varepsilon \to 0$. Hence, $(p, q) \in \overline{\mathbb{NA}}$.

II Proof of Proposition A.1

It is clear that $\Gamma_i(W_0, p, q)$ is non-empty because it contains the hand-to-mouth-plan, specifically $N_i = 0$, $c_{it}(s^t) = W_{it}(s^t)$ and, for $t > 0$, $W_{it}(s^t) = e_{it}(s^t)$. The constraint correspondence is convex-valued because it is defined by a system of weak linear inequalities. The closed graph property is also immediate because the correspondence is defined by weak inequalities, all jointly continuous in (W_0, p, q) and (c_i, N_i).[27] The only thing left to show is lower hemi-continuity.

II.1 The one-period ahead constraint correspondence

In this subsection, we fix some node (t, s^t) and establish the lower hemi-continuity of the one-period ahead constraint correspondence. Since the node is fixed, we suppress explicit reference to it to simplify notations. Instead, we use the "0" subscript for variables indexed by (t, s^t), and the "1" subscript, together with the argument s, for variables indexed by (t, s^t, s). With this in mind, we define the one-period ahead constraint correspondence as the set-valued function mapping any (W_0, p, q) to the corresponding one period-ahead constraint set, that is, the set of $(c_0, N_0, W_1) \in \mathbb{R}_+ \times \mathcal{M}_+ \times \mathbb{R}_+^S$ such that:

$$\begin{aligned}
&q_0 c_0 + \sum_s q_1(s) W_1(s) + \int p_0(\delta)\, dN_0(\delta) \\
=&q_0 W_0 + \sum_s q_1(s) e_1(s) + \sum_s \int \left[q_1(s)\delta_1(s) + p_1(\delta \mid s)\right] dN_0(\delta),
\end{aligned} \tag{56}$$

and, for all $s \in S$:

$$q_1(s) W_1(s) \geq q_1(s) e_1(s) + \int \theta_i(\delta) \left[q_1(s)\delta_1(s) + p_1(\delta \mid s)\right] dN_0(\delta). \tag{57}$$

Equipped with this definition, we obtain:

[27] For the joint continuity, recall from Corollary 15.7 in Aliprantis and Border (2006) that the "evaluation function" $(p, N) \mapsto \int p(\delta)\, dN(\delta)$ is jointly continuous in (p, N).

Lemma II.1 *The one-period ahead constraint correspondence is lower hemi-continuous in* $(W_0, p, q) \in \mathbb{R}_+ \times \mathbb{NA}$.

Consider some sequence $(W_0^\ell, p^\ell, q^\ell) \to (W_0, p, q)$ and some (c_0, N_0, W_1) in the one-period ahead constraint set given (W_0, p, q). To prove lower hemi-continuity, we must find a sequence $(c_0^\ell, N_0^\ell, W_1^\ell) \to (c_0, N_0, W_1)$, such that for all ℓ large enough, $(c_0^\ell, N_0^\ell, W_1^\ell)$ belongs to one-period ahead constraint set given $(W_0^\ell, p^\ell, q^\ell)$. We distinguish four mutually exclusive cases.

Case 1: $q_0\, c_0 > 0$.

Then we pick the sequence $(c_0^\ell, N_0^\ell, W_1^\ell)$ such that:

$$
\begin{aligned}
& N_0^\ell = N_0 \\
& q_1^\ell(s) W_1^\ell(s) = \max\left\{ q_1^\ell(s) e_1(s) + \int \theta_i(\delta) \left[q_1^\ell(s)\delta_1(s) + p_1(\delta \,|\, s) \right] \, dN_0(\delta), q_1^\ell(s) W_1(s) \right\} \qquad (58) \\
& q_0^\ell c_0^\ell + \int p_0(\delta) \, dN_0(\delta) + \sum_s q_1^\ell(s) W_1^\ell(s) \\
& \qquad = q_0^\ell W_0^\ell + \sum_s q_1^\ell(s) e_1^\ell(s) + \sum_s \int \left(q_1^\ell(s)\delta_1(s) + p_1^\ell(\delta \,|\, s) \right) \, dN_0(\delta).
\end{aligned}
$$

Since $q_1^\ell(s)W_1^\ell(s) \to q_1(s)W_1(s)$, it follows that $q_0^\ell c_0^\ell \to q_0 c_0$, and so is positive for all ℓ large enough. Since all q are strictly positive, it follows that $(c^\ell, N_0^\ell, W_1^\ell) \to (c, N_0, W_1)$.

Case 2: $q_0 c_0 = 0$ and $q_1(s)W_1(s) = q_1(s)e_1(s)$ for each s.

Then it follows from the incentive constraints that

$$
\int \theta_i(\delta) \left[q_1(s)\delta_1(s) + p_1(\delta \,|\, s) \right] \, dN_0(\delta) = 0.
$$

Therefore, $N_0(\Delta_0^+) = 0$, where Δ_0^+ is the set of trees with non-zero continuation dividend streams. Plugging this into the budget constraint shows that $W_0 = 0$. We then pick the sequence $c_0^\ell = W_0^\ell$, $N_0^\ell = N_0$ and $W_1^\ell(s) = e_1(s)$.

Case 3: $q_0 c_0 = 0$ and $q_1(\hat{s})W_1(\hat{s}) > q_1(\hat{s})e_1(\hat{s}) + \int \theta_i(\delta) \left[q_1(\hat{s})\delta_1(\hat{s}) + p_1(\delta \,|\, \hat{s}) \right] \, dN_0(\delta)$ for some $\hat{s}$.

Then we pick the sequence $c_0^\ell = c_0 = 0$ and $N_0^\ell = N_0$. For $s \neq \hat{s}$, we choose $W_1^\ell(s)$ according to (58). Then, we pick $W_1^\ell(\hat{s})$ so that the budget constraint (56) holds. By construction, $(c^\ell, N_0^\ell, W_1^\ell) \to (c, N_0, W_1)$. Moreover, for all ℓ, the budget constraint hold, as well as the incentive constraints for $s \neq \hat{s}$. But by our maintained assumption the incentive it holds strictly in the limit for $s = \hat{s}$, so it holds for ℓ large enough too.

Case 4: $q_0 c_0 = 0$, the incentive constraints bind for all s, and $q_1(\hat{s})W_1(\hat{s}) > q_1(\hat{s})e_1(\hat{s})$ for some $\hat{s}$.

In this case, because $q_1(\hat{s})W_1(\hat{s}) > q_1(\hat{s})e_1(\hat{s})$ but the incentive constraint binds for $\hat{s}$, it follows from the incentive constraint at $\hat{s}$ that

$$
\int \theta_i(\delta) \left[q_1(\hat{s})\delta_1(\hat{s}) + p_1(\delta \,|\, \hat{s}) \right] \, dN_0(\delta) > 0,
$$

so that $N_0(\Delta^+) > 0$. Now for some $\varepsilon^\ell \to 0$ to be determined later, pick $N_0^\ell = (1 - \varepsilon^\ell)N_0$. Pick $W_1^\ell(s)$ so that the incentive constraint binds:

$$q_1^\ell(s)W_1^\ell(s) = -\varepsilon^\ell \int \theta_i(\delta) \left[q_1^\ell(s)\delta_1(s) + p_1^\ell(\delta \,|\, s)\right] dN_0(\delta) + q_1^\ell(s)\hat{W}_1^\ell(s)$$

$$\text{where } q_1^\ell(s)\hat{W}_1^\ell(s) \equiv q_1^\ell(s)e_1(s) + \int \theta_i(\delta) \left[q_1^\ell(s)\delta_1(s) + p_1^\ell(\delta \,|\, s)\right] dN_0(\delta)$$

We now pick ε^ℓ and c_0^ℓ so that the bugdet constraint binds. After some algebra we obtain that, for the budget constraint to bind, ε^ℓ and c_0^ℓ must be chosen so that:

$$\begin{aligned}
&q_0^\ell c_0^\ell - \varepsilon^\ell \int \left(p_0^\ell(\delta) - \sum_s (1-\theta_i(\delta)) \left[q_1^\ell(s)\delta_1(s) + p_1^\ell(\delta \,|\, s)\right]\right) dN_0(\delta) \\
=&W_0^\ell + \sum_s q_1^\ell(s)e_1(s) + \int \left(\sum_s \left[q_1^\ell(s)\delta_1(s) + p_1^\ell(\delta \,|\, s)\right] - p_0(\delta)\right) dN_0(\delta) - \sum_s q_1^\ell(s)\hat{W}_1^\ell(s)
\end{aligned}$$

Since $\hat{W}_1^\ell(s) \to W_1(s)$, $q_0 c_0 = 0$, and since the budget constraint holds in the limit, it follows that the right-hand side goes to zero as $\ell \to \infty$. If we denote this right-hand side by η^ℓ, then we can set c_0^ℓ and ε^ℓ to

$$\begin{aligned}
&q_0 c_0^\ell = \max\{\eta^\ell, 0\} \\
&\varepsilon^\ell \int \left[p_0^\ell(\delta) - (1-\theta_i(\delta)) \sum_s \left(q_1^\ell(s)\delta_1(s) + p_1^\ell(\delta \,|\, s)\right)\right] dN_0(\delta) = \min\{-\eta^\ell, 0\}.
\end{aligned}$$

Notice that the integral multiplying ε^ℓ is bounded away from zero for ℓ large enough. Indeed,it converges to

$$\int \left[p_0(\delta) - (1-\theta_i(\delta)) \sum_s \left(q_1(s)\delta_1(s) + p_1(\delta \,|\, s)\right)\right] dN_0(\delta)$$

which is strictly positive for two reasons: first, $(p, q) \in \mathbb{NA}$ so that the integrand is strictly positive over Δ^+, and second $N_0(\Delta^+) > 0$ so that the integral is strictly positive as well. This ensures that ε^ℓ is well defined for ℓ large enough, and goes to zero as ℓ goes to infinity.

II.2 The multi-period constraint correspondence

The proof now follows by recursive application of Lemma II.1 in the previous subsection. Namely, consider some $(W_0, p, q) \in \mathbb{NA}$, some $(c, N) \in \Gamma_i(W_0, p, q)$. Let W denote the sequence of cash on hand associated with this plan. Consider any sequence $(W_0^\ell, p^\ell, q^\ell) \to (W_0, p, q)$. Lemma II.1 allows to construct sequences $c_0^\ell(s^0)$, $W_1^\ell(s^0, s_1)$, $N_0^\ell(s^0)$ converging to W_0, $c_0(s^0)$, $N_0(s^0)$ and $W_1(s^0, s_1)$. satisfying the one-period ahead budget constraint. Now repeat this step at each node (t, s^t) until reaching the second to last period, $T-1$.

III Proof of Proposition A.2

III.1 Unbounded constraint set

We first establish that:

Lemma III.1 *Take any* $(p,q) \in \overline{\mathbb{NA}} \setminus \mathbb{NA}$*. Then for any sequence* (p^ℓ, q^ℓ) *in* $\mathbb{NA}$ *converging to* (p,q)*, there exists some agent* i *and some sequence* $(c_i^\ell, N_i^\ell) \in \Gamma_i(W_0^\ell, p^\ell, q^\ell)$*, where* W_0^ℓ *is defined by* (9)*, such that* $\|c_i^\ell\| \to \infty$*.*

Lemma A.1 implies that, if $(p,q) \in \overline{\mathbb{NA}} \setminus \mathbb{NA}$, either

$$q_t(s^t) = 0 \tag{59}$$

for some (t, s^t), or there exists some agent i, some (t, s^t) and some $\delta \in \Delta_t^+(s^t)$, such that

$$p_t(\delta \,|\, s^t) = (1 - \theta_i(\delta)) \sum_s \left[q_{t+1}(s^t, s)\delta_{t+1}(s^t, s) + p_{t+1}(\delta \,|\, s^t, s)\right] \tag{60}$$

For what follows let us denote by

$$T_i^\ell \equiv \alpha_i \int \left[q_0^\ell(s^0)\delta_0(s^0) + p_0^\ell(\delta \,|\, s^0)\right] \, d\bar{N}(\delta)$$

the initial tree wealth of the agent. Notice that T_i^ℓ has a strictly positive limit as $\ell \to \infty$. Ineeed, for any price system $(p,q) \in \overline{\mathbb{NA}}$, iterating forward on the no-arbitrage condition (13) (with a weak inequality) starting at time $t = 0$, implies that

$$q_0(s^0)\delta_0(s^0) + p_0(\delta \,|\, s^0) \geq \sum_{(t,s^t)} (1 - \theta_i(\delta))^t q_t(s^t)\delta_t(s^t).$$

Integrating against the aggregate supply we obtain that:

$$\int \left[q_0(s^0)\delta_0(s^0) + p_0(\delta \,|\, s^0)\right] \, d\bar{N}(\delta) \geq \sum_{(t,s^t)} (1 - \theta_i(\delta))^t \int q_t(s^t)\delta_t(s^t) \, d\bar{N}(\delta) > 0, \tag{61}$$

given our assumption that the aggregate dividend is strictly positive in all state and since our normalization (34) ensures at least one of the consumption prices, $q_t(s^t)$, must be strictly positive. It thus follows that the value of every agent's initial tree wealth is strictly positive in the limit (p,q), and bounded away from zero near the limit.

Case 1. Suppose that the time-zero price of consumption is zero at some node, (t, s^t), $q_t(s^t) = 0$. Since all elements of the sequence (p^ℓ, q^ℓ) belong to $\mathbb{NA}$, $q_u^\ell(s^u) > 0$ for each node (u, s^u). Hence, an agent can save her initial tree wealth from node to node until (t, s^t), at which point she can consume the amount $c_{it}^\ell(s^t)$ solving $q_t^\ell(s^t)c_{it}^\ell(s^t) = q_t^\ell(s^t)e_{it}(s^t) + T_i^\ell$. At all other node, the agent can consume hand-to-mouth, $c_{iu}^\ell(s^u) = e_{iu}(s^u)$.[28] Since $q_t^\ell(s^t) \to 0$, and since T_i^ℓ remains bounded away from zero for all ℓ large enough, it follows that $c_{it}^\ell(s^t) \to \infty$.

Case 2. Now suppose that the price of consumption is strictly positive at all nodes but (60) holds for some i, some (t, s^t) and some $\delta \in \Delta_t^+(s^t)$. Then proceeding exactly as above the agent can choose cash on hand equal to $q_t^\ell(s^t)e_{it}(s^t) + T_i^\ell$ for node (t, s^t). At this point she can purchase an amount $n^\ell > 0$ of tree δ (formally, this portfolio is a discrete measure concentrated at δ) and partially finance the purchase by selling the tree pledgeable payoff. If

[28] The corresponding plan (c_i, N_i) is the following. Set $N_{iu}^\ell(s^u) = 0$ for all node. For all nodes $(u, s^u) \preceq (t, s^t)$, set $q_u^\ell(s^u)W_{iu}^\ell(s^u) = q_u^\ell(s^u)e_{iu}(s^u) + T_i^\ell$. For all other node, set $W_{iu}^\ell(s^u) = e_{iu}(s^u)$. For all nodes $(u, s^u) \neq (t, s^t)$, set $c_{iu}^\ell(s^u) = e_{iu}(s^u)$. For (t, s^t), pick $c_{it}^\ell(s^t)$ solving $q_t^\ell(s^t)c_{it}^\ell(s^t) = q_t^\ell(s^t)e_{it}(s^t) + T_i^\ell$.

the agent consumes hand to mouth at this node, $c_t^\ell(s^t) = e_{it}(s^t)$, then:

$$n^\ell \left(p_t^\ell(\delta \,|\, s^t) - (1-\theta_i(\delta)) \sum_s \left(q_{t+1}^\ell(s^t,s)\delta_{t+1}(s^t,s) + p_{t+1}^\ell(\delta \,|\, s^t,s) \right) \right) = T_i^\ell$$

$$q_{t+1}^\ell(s^t,s) W_{it+1}^\ell(s^t,s) = q_{t+1}^\ell(s^t,s) e_{t+1}(s^t,s) + n^\ell \theta_i(\delta) \left(q_{t+1}^\ell(s^t,s)\delta_{t+1}(s^t,s) + p_{t+1}^\ell(\delta \,|\, s^t,s) \right)$$

$$c_{it+1}^\ell(s^t,s) = W_{it+1}^\ell(s^t,s).$$

Clearly, $n^\ell \to \infty$. Since $\delta \in \Delta_t^+(s^t)$, it follows from the no-arbitrage relationship (37) holding with weak equality together with our maintained assumption that $q_t(s^t) > 0$ at all nodes, that the tree payoff is strictly positive in some state $\hat{s}$. It follows that $c_{t+1}^\ell(s^t,\hat{s}) \to \infty$.

III.2 Unbounded optimal demands

We now show that an unbounded constraint set implies an unbounded demand. Precisely, take any $(c_i^\ell, N_i^\ell) \in Z_i(p^\ell, q^\ell)$ and assume it is bounded in consumption. Use the previous lemma to generate another sequence $(\hat{c}_i^\ell, \hat{N}_i^\ell) \in \Gamma_i(W_{i0}^\ell, p^\ell, q^\ell)$, where W_{i0}^ℓ is defined according to (9), that is unbounded in consumption. Since $\|\hat{c}_i^\ell\| \geq 1$ for all ℓ large enough, it follows by convexity that, for all ℓ large enough, the plan

$$\left(1 - \frac{1}{\|\hat{c}_i^\ell\|}\right)(c_i^\ell, N_i^\ell) + \frac{1}{\|\hat{c}_i^\ell\|}(\hat{c}_i^\ell, \hat{N}_i^\ell)$$

belongs to the constraint set $\Gamma_i(W_{i0}^\ell, p^\ell, q^\ell)$, where W_{i0}^ℓ is defined according to (9). Moreover, since the norm of $\hat{c}_i^\ell / \|\hat{c}_i^\ell\|$ is equal to one, at least one of its coordinate must remain bounded away from zero. It then follows by continuity that the above convex combination must yield strictly higher intertemporal utility than (c_i^ℓ, N_i^ℓ) for all ℓ large enough, and we have reached a contradiction.

IV Proof of Proposition A.3

Necessity. Suppose (c_i, N_i) solves the agent's problem together with some cash-on-hand plan W_i. Since utility is strictly increasing, (c_i, N_i), it also solves the "relaxed" problem in which all sequential budget constraints are only required to hold with a weak inequality. Since the constraint set is defined by a finite number of inequality constraints, the associated positive cone has interior points. Since the agent starts with strictly positive tree wealth, it can guarantee strictly positive cash-on-hand at all nodes, so there is some plan for consumption, tree portfolio, and cash-on-hand such that all constraints hold with strict inequalities. It then follows from Theorem 1, chapter 8.3 in Luenberger (1969) that there exists positive multipliers λ and μ, as shown in the Proposition, such that the associated Lagrangian is maximized at (c_i, N_i, W_i), and such that the associated complementarity slackness conditions hold. The result follows.

Sufficiency. This follows by a standard optimality-verification argument.

From Proposition A.3 to the first-order conditions shown in the text. The first-order condition (16) and (17) shown in the text obtain by re-defining $\lambda_{it}(s^t) \equiv \hat{\lambda}_{it}(s^t) q_t(s^t)$ and $\mu_{it}(s^t) \equiv \hat{\mu}_{it}(s^t) q_t(s^t)$. The first-order condition (20) requires some algrebraic manipulations. First, (41) implies that

$$\sum_s \left(1 - \frac{\theta_i(\delta)\hat{\mu}_{it+1}(s^t,s)}{\hat{\lambda}_{it}(s^t)}\right)\left[q_{t+1}(s^t,s)\delta_{t+1}(s^t,s) + p_{t+1}(\delta \,|\, s^t,s)\right] \leq p_t(\delta \,|\, s^t),$$

with an equality if $dN_{it}(\delta \,|\, s^t) > 0$. Now dividing through both sides by $q_t(s^t)$ and keeping in mind that $Q_{t+1}(s^t,s) = q_{t+1}(s^t,s)/q_t(s^t)$, this inequality writes:

$$\sum_s \left(1 - \frac{\theta_i(\delta)\hat{\mu}_{it+1}(s^t,s)}{\hat{\lambda}_{it}(s^t)}\right) Q_{t+1}(s^t,s)\left[\delta_{t+1}(s^t,s) + P_{t+1}(\delta \,|\, s^t,s)\right] \leq P_t(\delta \,|\, s^t),$$

The term in parenthesis can be then rewritten:

$$\begin{aligned}
1 - \frac{\theta_i(\delta)\hat{\mu}_{it+1}(s^t,s)}{\hat{\lambda}_{it}(s^t)} &= 1 - \theta_i(\delta) + \theta_i(\delta)\frac{\hat{\lambda}_{it}(s^t) - \hat{\mu}_{it+1}(s^t,s)}{\hat{\lambda}_{it}(s^t)} \\
&= 1 - \theta_i(\delta) + \theta_i(\delta)\frac{\hat{\lambda}_{it+1}(s^{t+1})}{\hat{\lambda}_{it}(s^t)} \\
&= 1 - \theta_i(\delta) + \theta_i(\delta)\frac{\lambda_{it+1}(s^{t+1})}{Q_{t+1}(s^t,s)\lambda_{it}(s^t)},
\end{aligned}$$

where the equality on the second line follows from (39), and the equality on the third line follows from the definition of $\lambda_{it}(s^t)$ and $\mu_{it}(s^t)$. The first-order condition (20) follows.

V Proof of Proposition A.4

Using our basis notation, the sequential budget constraint of agent i writes, for all $t < T$

$$\begin{aligned}
&q_t(s^t)c_{it}(s^t) + \sum_{s'} q_{t+1}(s^t,s)W_{it+1}(s^t,s) \\
=&q_t(s^t)W_{it}(s^t) + \sum_s q_{t+1}(s^t,s)e_{it+1}(s^t,s) + \int b_t(\delta \,|\, s^t)\, dN_{it}(\delta \,|\, s^t).
\end{aligned} \tag{62}$$

Adding up all the constraint until $T-1$, as well as the time-T constraint $W_{iT}(s^T) = c_{iT}(s^T)$, all the $W_{it}(s^t)$ cancel out except for the first one, $W_{i0}(s^0) = e_{i0}(s^0) + \alpha_i \int \left(q_0(s^0)\delta_0(s^0) + p_0(\delta \,|\, s^0)\right) d\bar{N}(\delta)$. We thus obtain the intertemporal budget constraint:

$$\sum_{(t,s^t)} q_t(s^t)c_{it}(s^t) = \alpha_i \int \left(q_0(s^0)\delta_0(s^0) + p_0(\delta \,|\, s^0)\right) d\bar{N}(\delta) + \sum_{(t,s^t)} q_t(s^t)e_{it}(s^t) + \sum_{(t,s^t),t<T} \int b_t(\delta \,|\, s^t)\, dN_{it}(\delta \,|\, s^t).$$

Now recall that, by definition of the basis:

$$p_t(\delta \,|\, s^t) = \sum_{s'} \left(q_{t+1}(s^t,s')\delta_{t+1}(s^t,s') + p_{t+1}(\delta \,|\, s^t,s')\right) - b_t(\delta \,|\, s^t)$$

for all (t, s^t), $t < T$. Iterating forward on this equation starting at the initial node $(0, s^0)$, we obtain that:

$$q_0(s^0)\delta_0(s^0) + p_0(\delta \mid s^0) = \sum_{(t,s^t)} q_t(s^t)\delta_t(s^t) - \sum_{(t,s^t),t<T} b_t(\delta \mid s^t),$$

that is, tree prices are equal to the present value of all of their future dividends, net of the sum of all their bases. Plugging back into the intertemporal budget constraint we obtain:

$$\sum_{(t,s^t)} q_t(s^t)c_{it}(s^t) = \sum_{(t,s^t)} q_t(s^t)\left(e_{it}(s^t) + \alpha_i \int \delta_t(s^t)\, d\bar{N}(\delta)\right) + \sum_{(t,s^t),t<T} \int b_t(\delta \mid s^t)\, \left(dN_{it}(\delta \mid s^t) - \alpha_i d\bar{N}(\delta)\right)$$

and the result follows by adding up these intertemporal constraints across all i and using that $\sum_i \alpha_i = 1$.

V.1 Lower hemi-continuity for the multi-period constraint set bounded by B

To show lower hemi-continuity at (W_0, p, q) consider any $(c, n) \in \Gamma_i(W_0, p, q) \cap B$ and any $(W_0^\ell, p^\ell, q^\ell) \to (W_0, p, q)$. Since the correspondence $\Gamma_i(W_0, p, q)$ is lower hemi-continuous, we already know that there is a sequence $(c^\ell, n^\ell) \in \Gamma_i(W_0^\ell, p^\ell, q^\ell)$ converging to (c, n). But this sequence may lie outside B. To bring each element of the sequence inside B, take a convex combination of (c^ℓ, n^ℓ) with the hand-to-mouth plan, which we denote $(\hat{c}^\ell, \hat{n}^\ell)$: $\hat{n}_t(s^t) = 0$, $\hat{c}_0^\ell(s^0) = W_0^\ell(s^0)$, $\hat{c}_t(s^t) = \hat{W}_t^\ell(s^t) = e_t(s^t)$ for $t > 0$. Since $(\hat{c}^\ell, \hat{n}^\ell) \in B$, there exists $\lambda \in [0, 1]$ such that:

$$\lambda(\hat{c}^\ell, \hat{n}^\ell) + (1 - \lambda)(c^\ell, n^\ell) \in B.$$

Let λ^ℓ be the smallest λ such that this is the case. Recall that, for $(\hat{c}, \hat{n})$, the right-hand-side inequalities defining B, (44) and (45), hold strictly. Together with the fact that $(\hat{c}^\ell, \hat{n}^\ell) \to (\hat{c}, \hat{n})$ and $(c^\ell, n^\ell) \to (c, n) \in B$, we obtain that, for any $\lambda > 0$, $\lambda(\hat{c}^\ell, \hat{n}^\ell) + (1 - \lambda)(c^\ell, n^\ell)$ satisfies these right-hand-side inequality strictly for all ℓ large enough. Therefore, $\lambda^\ell \leq \lambda$ for all ℓ large enough. Since this is true for all $\lambda > 0$, we obtain that $\lambda^\ell \to 0$ and have found the desired sequence.

V.2 Proof of Proposition B.1

The fixed point problem has a solution. By an application of the Theorem of the Maximum (see Theorem 3.6 in Stokey and Lucas, 1989), this correspondence is compact valued and upper hemi-continuous.[29] It is also convex valued because the constraints sets are convex and the objective concave. Hence, an application of the Kakutani fixed point theorem (see Theorem M.I.2. p.953 in Mas-Colell, Whinston and Green, 1995) shows that it has a fixed point, $(p^\varepsilon, q^\varepsilon, (c_i^\varepsilon, n_i^\varepsilon)_{i \in I})$. By Walras Law stated in Proposition A.4, the maximized price player objective must be equal to zero.

The B constraints are not binding. Notice that, given any q, it is always feasible for the price player to choose tree prices with zero bases, $\mathbb{b}(p, q^\varepsilon) = 0$. Therefore, the price player's value is weakly negative when she sets

[29] When $u_i(c)$ is unbounded below, the Theorem does not apply directly because the domain of the utility function is $(0, \infty)$ instead of $[0, \infty)$, so the constraint set is not compact. But one can restrict the domain to consumptions that are bounded away from zero by some constant. Indeed, since aggregate dividends are strictly positive at all nodes and Arrow security prices add up to one, agents' initial wealth can be bounded away from zero (see equation (61)). Therefore, purchasing Arrow securities only, whose prices are bounded above by 1, the agent can guarantee herself positive consumption at all nodes: formally, there exists some (c_i, N_i), such that $c_{it}(s^t) > 0$ at all (t, s^t) and $N_i = 0$, which is budget feasible and incentive compatible for all prices $(p^\varepsilon, q^\varepsilon)$. This places a lower bound on the agent's maximum attainable utility and, when utility is unbounded below, a strictly positive lower bound on the agent's consumption. Hence we can restrict the domain, as claimed.

$q_t(s^t) = \bar{q}$, for some constant $\bar{q}$ such that consumption prices add up to one, and chooses p such that $\mathbb{b}(p,q) = 0$. This implies that:

$$\sum_{(t,s^t)} \sum_i c_{it}(s^t) \leq \sum_{(t,s^t)} \omega_t(s^t),$$

so that, at the fixed point, the constraint (44) is slack for consumption.

Then, we argue that the constraint (45) is slack for tree holdings as well. This is evident if if $\sum_i n_{ikt}(s^t) \leq \bar{N}_k$. If $\sum_i n_{ikt}(s^t) > \bar{N}_k$ for some tree k and node (t,s^t), then $\mathbb{b}_{kt}(s^t \,|\, p^\varepsilon, q^\varepsilon) = 0$, otherwise, the price player could increase its value by reducing the basis of tree k at node (t,s^t) to zero, and keep all other bases the same. To see that such reduction in basis is consistent with the no-arbitrage conditions (i.e., the corresponding price system remains in the constraint set of the price player) let us consider any price system $(p,q) \in \mathbb{NA}^\varepsilon$ and show that the price system remains in $\mathbb{NA}^\varepsilon$ if one reduces the basis of any specific tree to zero at any node (t,s^t). Indeed, consider the tree prices $\hat{p}$ obtained by inverting the basis formula

$$\hat{p}_t(s^t) = \sum_{s'} \left(q_{t+1}(s^t,s')\delta_{t+1}(s^t,s') + \hat{p}_{t+1}(s^t,s') \right) - \hat{b}_t(s^t). \tag{63}$$

from $\hat{p}_T(s^T) = 0$, setting $\hat{b}_{kt}(t,s^t) = 0$, but otherwise keeping all other bases the same, $\hat{b}_{ju}(s^u) = b_{ju}(s^u)$ for $j \neq k$ or $(u,s^u) \neq (t,s^t)$. For all trees $j \neq k$, the tree prices remain the same. For tree k, the price is the same at all nodes $(u,s^u) \succ (t,s^t)$, and it increases strictly by $b_{kt}(s^t)$ at all nodes $(u,s^u) \preceq (t,s^t)$. But this relaxes the constraint

$$\begin{aligned} & \hat{p}_{ku}(s^u) \geq (1 - \theta_{ik} + \varepsilon) \sum_{s'} \left(q_{u+1}(s^u,s')\delta_{ku+1}(s^u,s') + \hat{p}_{ku+1}(s^u,s') \right) \\ \Leftrightarrow \quad & \hat{b}_{ku}(s^u) \leq (\theta_{ik} - \varepsilon) \sum_{s'} \left(q_{u+1}(s^u,s')\delta_{ku+1}(s^t,s') + \hat{p}_{ku+1}(s^t,s') \right) \end{aligned}$$

since $\hat{b}_{ku}(u,s^u) \leq b_{ku}(s^u)$ and $\hat{p}_{ku}(s^u) \geq p_{ku}(s^u)$.

Now, since $\mathbb{b}_{kt}(s^t \,|\, p,q) = 0$, then the agent can reduce its holdings of tree k arbitrarily, keep the same budget, and relax her incentives constraint. Hence, (45) is not binding either.

For ε small enough, the constraint (46) and (47) are not binding. The finding that both (44) and (45) are slack implies, together with the the concavity of the objective and the convexity of the constraint set, that $(c_i^\varepsilon, n_i^\varepsilon)$ solves the agent's problem given $(p^\varepsilon, q^\varepsilon)$, but unconstrained by B, that is, subject to $(c_i, n_i) \in \Gamma_i(W_0^\varepsilon, p^\varepsilon, q^\varepsilon)$ only, where W_0^ε is defined by (9). Moreover, all the c_i^ε are uniformly bounded. Now consider any sequence $\varepsilon^\ell \to 0$, an associated sequence of fixed points (p^ℓ, q^ℓ) and $(c_i^\ell, n_i^\ell)_{i \in I}$. Given that all of these sequences are bounded we can extract convergent subsequences (but keep the same notation). By Proposition A.2, the limit price cannot belong to $\overline{\mathbb{NA}} \setminus \mathbb{NA}$ otherwise the associated consumption sequence of some agent would be unbounded. Hence, the limit of the price sequence belongs to $\mathbb{NA}$. This implies that, for ℓ large enough, the constraint (46) is not binding, and the constraint (47) is not binding either for trees with non-zero continuation dividends.

Tree-market clearing. For a tree with zero continuation dividend at (t,s^t), the no-arbitrage conditions imply that current and future prices are zero. This makes the tree is irrelevant in all current and future budget and incentive constraints, and implies that any demand is optimal. Therefore, we can pick the demand so as to clear the market.

For a tree with non-zero continuation dividend, consider the corresponding term in the price player's objective

$$-b_{kt}(s^t)\left(\sum_i n_{ikt}(s^t) - \bar{N}_k\right).$$

Now, we have shown that, for ε small enough, the constraint (47) of the price player is not binding for all $(u, s^u) \preceq (t, s^t)$. This means that the price player can, without violating its no-arbitrage constraints, increase slightly the basis of the tree at time (t, s^t) by reducing $p_{kt}(s^t)$, and keep all other bases the same by corresponding reductions in $p_{ku}(s^u)$ at $(u, s^u) \prec (t, s^t)$. Since this cannot increase the price player's objective, it follows that $\sum_i n_{ikt}(s^t) \geq \bar{N}_k$.

If, at node (t, s^t), the basis of tree k is strictly positive, then by the same argument used just above to show that the B constraint is not binding, the price player can, without violating its no-arbitrage constraints, reduce the basis $b_{kt}(s^t)$ and keep all other bases the same. Since this cannot improve the price player's objective, we obtain that $\sum_i n_{ikt}(s^t) \leq \bar{N}_k$, and we are done.

If, at node (t, s^t), the basis of tree k is zero, then reducing the demand of the tree leaves the budget constraint the same, and relaxes the incentive constraint. Hence we can adjust the demand of each agent in such a way that $\sum_i n_{ikt}(s^t) = \bar{N}_k$.

Consumption-market clearing. For ℓ large enough, the constraint (46) is not binding, and the constraint (47) is not binding for trees with non-zero continuation dividend. Hence, it is feasible for the price player to increase and decrease slightly any two coordinates of $q_t^\ell(s^t)$ and keep all the bases the same. Since this cannot increase the price player's objective, we conclude that $\sum_i c_{it}^\ell(s^t) - \omega_t(s^t)$ is constant across all nodes.

Now recall that, at the fixed point, Walras Law implies that the price player's objective must be zero. Since we know that the tree market clears, this writes:

$$\sum_{(t,s^t)} q_t^\ell(s^t)\left[\sum_i c_{it}^\ell(s^t) - \omega_t(s^t)\right] = 0.$$

But since $\sum_i c_{it}^\ell(s^t) - \omega_t(s^t)$ is constant across all nodes, it follows that it must be equal to zero.

V.3 Proof of Lemma B.1

Proceeding as in Proposition A.3, we obtain that the first-order conditions with respect to n, for the economy with finitely many trees write:

$$v_{ikt}(s^t) - p_{kt}(s^t) \leq 0, \text{ with " = " if } n_{ikt}(s^t) > 0,$$

where

$$v_{ikt}(s^t) \equiv \sum_s \left(1 - \frac{\theta_i(\delta_k)\mu_{it+1}(s^t, s)}{\lambda_{it}(s^t)}\right)\left[q_{t+1}(s^t, s)\delta_{kt+1}(s^t, s) + p_{kt+1}(s^t, s)\right].$$

Now given q and the Lagrange multiplier's λ and μ, let us define the functions $\hat{p}_t(\delta \mid s^t)$ and $\hat{v}_{it}(\delta \mid s^t)$ and as follows, backwards from time $T-1$:

$$\hat{p}_t(\delta \mid s^t) = \max_i \hat{v}_{it}(\delta \mid s^t),$$

where

$$\hat{v}_{it}(\delta \,|\, s^t) = \sum_s \left(1 - \frac{\theta_i(\delta)\mu_{it+1}(s^t,s)}{\lambda_{it}(s^t)}\right)\left[q_{t+1}(s^t,s)\delta_{t+1}(s^t,s) + \hat{p}_{t+1}(\delta \,|\, s^t,s)\right],$$

with the convention that $p_T = 0$.

By construction, $\hat{p}_t(\delta_k \,|\, s^t) = p_{kt}(s^t)$. Moreover, it is clear that the first-order necessary and sufficient conditions continue to hold when agents can choose any tree in Δ. Finally, the price function be viewed as a jointly continous function of $\delta \in \Delta$ and of the finitely many multipliers ratios $\mu_{it+1}(s^t,s)/\lambda_{it}(s^t) \in [0,1]$, for all i, all $t \geq 0$, all $s^t \in S^t$, and $s \in S$. This function is uniformly continuous since its domain is compact. This implies that, when viewed as function of δ only, the price function belongs to an equi-continuous family.

VI Proof of Corollary 1

Suppose we allow short-selling of trees subject to the incentive constraint (15) as explained prior to Corollary 1. Write the sequential budget constraints as in equation (3) (i.e. do not make the cash-on-hand change of variable), using deflated time-zero prices (p,q). Assume that the pledgeability parameter depends on assets and agents, $\theta_i(s)$ for Arrow securities and $\theta_i(\delta)$ for trees held by agent i. Proceeding as in Section A.3, the first-order necessary and sufficient conditions with respect to $c_{it}(s^t)$, $a^+_{it+1}(s^t,s)$, $a^-_{it+1}(s^t,s)$, $N^+_{it}(s^t)$, and $N^-_{it}(s^t)$, write:

$$\beta^t \pi_t(s^t) u_i'(c_t(s^t)) \leq \hat{\lambda}_{it}(s^t) q_t(s^t) \text{ with} \text{``}=\text{''} \text{ if } c_{it}(s^t) > 0 \tag{64}$$

$$\hat{\lambda}_{it+1}(s^t,s) + (1-\theta_i(s_t))\hat{\mu}_{it+1}(s^t,s) \leq \hat{\lambda}_{it}(s^t) \text{ with ``}=\text{'' if } a^+_{it+1}(s^t,s) > 0 \tag{65}$$

$$\hat{\lambda}_{it+1}(s^t,s) + \hat{\mu}_{it+1}(s^t,s) \geq \hat{\lambda}_{it}(s^t) \text{ with ``}=\text{'' if } a^-_{it+1}(s^t,s) > 0 \tag{66}$$

$$\int \left[\hat{v}^+_{it}(\delta \,|\, s^t) - p_t(\delta \,|\, s^t)\right] dM(\delta) \leq 0 \text{ for all } M \in \mathcal{M}_+ \text{ with ``}=\text{'' if } M = N^+_{it}(s^t) \tag{67}$$

$$\int \left[p_t(\delta \,|\, s^t) - \hat{v}^-_{it}(\delta \,|\, s^t)\right] dM(\delta) \leq 0 \text{ for all } M \in \mathcal{M}_+ \text{ with ``}=\text{'' if } M = N^-_{it}(s^t) \tag{68}$$

where $\hat{v}^+_{it}(\delta \,|\, s^t)$ and $\hat{v}^-_{it}(\delta \,|\, s^t)$ are the agent's private valuations for long and short positions:

$$\hat{v}^+_{it}(\delta \,|\, s^t) = \sum_s \frac{\hat{\lambda}_{it+1}(s^t,s) + (1-\theta_i(\delta))\hat{\mu}_{it+1}(s^t,s)}{\hat{\lambda}_{it}(s^t)}\left[q_{t+1}(s^t,s)\delta_{t+1}(s^t,s) + p_{t+1}(\delta \,|\, s^t,s)\right] \tag{69}$$

$$\hat{v}^-_{it}(\delta \,|\, s^t) = \sum_s \frac{\hat{\lambda}_{it+1}(s^t,s) + \hat{\mu}_{it+1}(s^t,s)}{\hat{\lambda}_{it}(s^t)}\left[q_{t+1}(s^t,s)\delta_{t+1}(s^t,s) + p_{t+1}(\delta \,|\, s^t,s)\right], \tag{70}$$

together with the complementary slackness condition for the incentive constraint:

$$\hat{\mu}_{it+1}(s^t,s)\Bigg\{\int (1-\theta_i(\delta))\left[q_{t+1}(s^t,s)\delta_{t+1}(s^t,s) + p_{t+1}(\delta \,|\, s^t,s)\right] dN^+_{it}(\delta \,|\, s^t) + (1-\theta_i(s))a^+_{it+1}(s^t,s) \tag{71}$$

$$- \int \left[q_{t+1}(s^t,s)\delta_{t+1}(s^t,s) + p_{t+1}(\delta \,|\, s^t,s)\right] dN^-_{it}(\delta \,|\, s^t) - a^-_{it+1}(s^t,s)\Bigg\} = 0. \tag{72}$$

Now, start from Theorem 1 and consider an equilibrium of the economy in which trees cannot be sold short,

(p, q, c, N). Let a denote the implied Arrow security positions of the agents. Let $N^+ \equiv N$, $N^- \equiv 0$, $a^+ \equiv \max\{a, 0\}$ and $a^- \equiv \max\{-a, 0\}$. We seek to show that $(p, q, c, N^+, N^-, a^+, a^-)$ is an equilibrium of the economy in which short-selling of tree is allowed, subject to the incentive constraint (15). Since the market-clearing conditions hold by construction, it is sufficient to show that $(c_i, N_i^+, N_i^-, a_i^+, a_i^-)$ satisfies the first-order sufficient conditions (64)-(72), given the multipliers found in the first-order necessary conditions (38)-(41) in Proposition A.3.

- The first-order condtition (64) holds because it is identical to (38).
- Notice that, because the incentive constraint is not binding when $a_{it+1}(s^t, s) > 0$, we have that $\hat{\mu}_{it+1}(s^t, s) = 0$ whenever $a_{it+1}(s^t, s) > 0$. Hence, the first-order condition with respect to $W_{it+1}(s^t, s)$, (39), implies the first-order conditions (65) and (66).
- The first-order condition (67) holds because it is identical to (66), given that (39) implies that $\hat{\lambda}_{it}(s^t) = \hat{\lambda}_{it+1}(s^t, s) + \hat{\mu}_{t+1}(s^t, s)$.
- Next recall that $\hat{\lambda}_{it}(s^t) = \hat{\lambda}_{it+1}(s^t, s) + \hat{\mu}_{t+1}(s^t, s)$, so

$$v_{it}^-(\delta \,, | \, s^t) = \sum_s \left[q_{t+1}(s^t, s)\delta_{t+1}(s^t, s) + p_{t+1}(\delta \,|\, s^t, s) \right].$$

  But we know that trees are priced below replicating portfolios of Arrow securities, so $v_{it}^-(\delta \,|\, s^t) \geq p_t(\delta \,|\, s^t)$, implying the first-order condition (68).
- Finally, complementary slackness condition holds by construction.

VII Proof of Lemma D.1

For sufficiency, start from a frictionless market equilibrium (q, c) and define the tree price function by the recursion:

$$p_t(\delta \,|\, s^t) = \sum_s q_{t+1}(s^t, s) \left[\delta_{t+1}(s^t, s) + p_{t+1}(\delta \,|\, s^t, s) \right], \tag{73}$$

for all nodes (t, s^t), $t < T$, with the convention that $p_T(s^T) = 0$. Define as well the cash-on-hand of agent i at node (t, s^t):

$$q_t(s^t) \left(W_{it}(s^t) - e_{it}(s^t) \right) = \sum_{(u, s^u) \succeq (t, s^t)} q_u(s^u) \left(c_{iu}(s^u) - e_{iu}(s^u) \right). \tag{74}$$

Let N denote the tree allocation in equation (48). We now verify that (p, q, c, N) is an equilibrium when agents are subject to incentive constraints. First, by construction, all markets clear. Second, $U_i(c_i)$ is an upper bound for the agent maximum attainable utility, when she is subject to incentive constraints in addition to budget constraint. But, together with (74), (48) ensures that the plan (c_i, N_i) is budget feasible and incentive compatible, so that this upper bound is attained. Hence, (c_i, N_i) is optimal for each agent given the price system (p, q).

Next, we turn to the proof of necessity. Suppose that (q, c) is IC implementable. That is, there exists an equilibrium $(\hat{p}, \hat{q}, \hat{c}, \hat{N})$ with incentive constraints such that $\hat{q} = q$ and $\hat{c} = c$. Then, comparing the first-order condition in the complete market equilibrium and in the corresponding equilibrium with incentive constraints shows that the multiplier on the incentive constraint is equal to zero, and as a result the no-arbitrage condition (73) holds. This implies that cash-on-hand can be written as (74), and that (48) holds with $N = \hat{N}$.

VIII Proof of properties of the zero-collateral equilibrium

In this appendix we study properties of the zero-collateral equilibrium, that is an equilibrium in the special case when $\bar{N}(\Delta) = 0$. First, we establish existence and uniqueness. Second, we establish continuity: equilibria can be made arbitrarily close to the zero collateral equilibrium, as long as $\bar{N}$ is small enough. This property is very convenient: it means that the zero-collateral equilibrium, which is straightforward to characterize, approximates all equilibria when $\bar{N}(\Delta) \simeq 0$.

Existence and uniqueness. Existence is straightforward but is not covered by our main Theorem, because an economy with zero collateral does not satisfy our assumption (1) that aggregate dividend are strictly positive at all nodes. Uniqueness is specific to this case and implies a continuity property: equilibria converge to the unique zero-collateral equilibrium as $\bar{N}(\Delta) \to 0$.

Proposition VIII.1 *Suppose that $e \gg 0$ and $\bar{N} = 0$. Then, in equilibrium, agents consume their labor endowment, $c_{it}(s^t) = e_{it}(s^t)$ and $N_i(s^t) = 0$ for all agents i and nodes (t, s^t). Moreover, an equilibrium price system is given by $q_0(s^0) = 1$,*

$$\frac{q_{t+1}(s^t, s)}{q_t(s^t)} = \max_i \beta \pi_{t+1}(s \mid s^t) \, \frac{u_i'(e_{it+1}(s^t, s))}{u_i'(e_{it}(s^t))}$$

$$p_t(\delta \mid s^t) = \max_i \sum_s \left(1 - \theta \frac{\hat{\mu}_{it+1}(s^t, s)}{\hat{\lambda}_{it}(s^t)}\right) \left[q_{t+1}(s^t, s)\delta_{t+1}(s^t, s) + p_{t+1}(\delta \mid s^t, s)\right]$$

where $\hat{\lambda}_{it}(s^t) = \beta^t \pi_t(s^t) \, u_i'(e_{it}(s^t))/q_t(s^t)$, and $\hat{\mu}_{it+1}(s^t, s) = \hat{\lambda}_{it}(s^t) - \hat{\lambda}_{it+1}(s^t, s)$.

We first show that the proposed price system and allocation is an equilibrium. Indeed, it is obvious that the proposed allocation clears market and that, given the proposed price system, it satisfies the budget and incentive constraints of each agent. Optimality follows from the first-order conditions of Proposition A.3.

Next, we turn to uniqueness of the equilibrium allocation. Consider any equilibrium (p, q, c, N). Then, since $\bar{N} = 0$, we must have that $N_i = 0$ as well. The incentive constraints for agent i then implies that $W_{it}(s^t) \geq e_{it}(s^t)$ for all $t \geq 1$. But notice that, in equilibrium, at all nodes, $\sum_i q_t(s^t) W_{it}(s^t) = \sum_i q_t(s^t) e_{it}(s^t) + \int \left[q_t(s^t)\delta_t(s^t) + p_t(\delta \mid s^t)\right] \, d\bar{N}(\delta)$.[30] This implies that $W_{it}(s^t) = e_{it}(s^t)$ at all $t \geq 1$. The definition of time-zero cash-on-hand also implies that $W_{i0}(s^0) = e_{i0}(s^0)$ at $t = 0$. The sequential budget constraint then implies that $c_{it}(s^t) = e_{it}(s^t)$.

Continuity. Now we argue that, the zero-collateral equilibrium identified above approximates all equilibria as $\bar{N} \to 0$.

Proposition VIII.2 *Fix some endowment of labor income $e \gg 0$ and some sequence of tree supplies $\bar{N}^\ell$ such that $\bar{N}^\ell(\Delta) \to 0$, satisfying our maintained assumption (1) that aggregate dividend is strictly positive at all nodes. Consider any associated sequence of equilibria $(p^\ell, q^\ell, c^\ell, N^\ell)$, and the corresponding sequence of optimal payoff sets, X^ℓ, and assume that the tree pricing equation (26) holds for all trees in Δ. Let (p, q, c, N) denote the zero-collateral equilibrium identified in Proposition VIII.1, and X the corresponding optimal payoff sets. Then $(p^\ell, q^\ell, c^\ell, N^\ell) \to (p, q, c, N)$. Moreover, for any sequence of payoffs $x^\ell \in X^\ell_{jt}(s^t)$, if $x^\ell \to x$, then $x \in X_{jt}(s^t)$.*

[30] In the last period, $t = T$, this is simply a restatement of market clearing since $W_T = c_T$ and $p_T = 0$. Suppose that this result is true at $t + 1 \leq T$. Then, if we add all the sequential budget constraints (32) across all agents and use our induction hypothesis, we obtain:

$$\sum_i q_t(s^t) c_{it}(s^t) + \sum_i \int p_t(\delta \mid s^t) \, dN_{it}(\delta \mid s^t) = \sum_i q_t(s^t) W_{it}(s^t),$$

and the result follows from the market clearing condition for consumption and trees at node (t, s^t).

Let us first extract a subsequence converging to some $(\hat{p}, \hat{q}, \hat{c}, \hat{N})$. Since $\bar{N}^\ell \to 0$, it follows that $N_i^\ell \to 0$ for all i and, therefore, $\hat{N} = 0$. Next, after passing to the limits in budget and incentive constraints, and noting that $\hat{q} > 0$,[31] we can use the same argument as in the proof of uniqueness in Proposition VIII.1 to obtain that $\hat{c} = c = e$. Finally, given that $e_{it}(s^t) > 0$, we can pass to the limit in the first-order conditions and obtain that $\hat{p} = p$ and $\hat{q} = q$.

For the second result, let $Y_{it}(s^t)$ denote the intersection of the optimal payoff set of agent i at node (t, s^t) with the simplex. It solves the optimization problem:

$$Y_{it}(s^t) = \arg\max \left\{ Q_{it+1}(s^t) \cdot x - P_t(x \,|\, s^t) \right\},$$

with respect to $x \in \mathbb{R}_+^S$ such that $\sum x(s) = 1$ and where, with some abuse of notation $P(x \,|\, s^t) \equiv \max_j Q_{jt+1} \cdot x$ is the price of payoff x. Since the objective of this optimization problem is continuous in the private valuations Q_j, Berge Theorem implies that its arg max is upper hemi continuous in the private valuations. Now consider any sequence $x^\ell \in X_{it}^\ell(s^t)$ such that $x^\ell \to x$. If $x = 0$, then the result follows because 0 belongs to all payoff sets. If $x \neq 0$, then the sequence $y^\ell = x^\ell / \sum x^\ell(s)$ belongs to the intersection of the optimal payoff set with the simplex, $Y_{it}^\ell(s^t)$, which are upper-hemi continuous in private valuations. Moreover, by the continuity result we just established above, $Q_i^\ell \to Q_i$ for all i. Therefore, we obtain that $x / \sum x(s)$ belongs to the limit intersection, $Y_{it}(s^t)$ and, by implication, that x belongs to the limit optimal payoff set, $X_{it}(s^t)$.

IX Optimal payoff sets and power-diagrams

Power diagrams: a definition. Take some finite collection of S-dimensional vectors $a = (a_1, \ldots, a_I)$ and scalars $b = (b_1, \ldots, b_I)$. Then, *the power diagram generated by* (a, b) is the collection of sets:

$$Y_i \equiv \left\{ x \in \mathbb{R}^S \,:\, a_i \cdot x + b_i \geq a_j \cdot x + b_j \text{ for all } j \right\}.$$

It is clear from the definition that any $x \in \mathbb{R}^S$ must belong to some Y_i, that is, a power diagram covers $\mathbb{R}^S$. Aurenhammer (1987a,b) has shown that any covering generated by a power diagram can be viewed as the projections of a $S+1$ dimensional convex polyhedron on $\mathbb{R}^S$, and vice versa. Indeed, consider the polyhedron defined as:

$$\left\{ (x, y) \in \mathbb{R}^S \times \mathbb{R} \,:\, y \geq \max_j \{a_j \cdot x + b_j\} \right\}.$$

Then it is clear that Y_i can be viewed as the face of the polyhedron defined by $y = a_i \cdot x + b_i$ and $y \geq a_j \cdot x + b_j$ for all j.

Any collection of optimal payoff sets map into some $S-1$ dimensional power diagram. Since optimal payoff sets are cones, they are entirerly determined by their intersection with the $S-1$ dimensional simplex. Moreover, consider any x in the $S-1$-dimensional simplex, that is such that $x(s) \geq 0$ and $\sum_{s=1}^S x(s) = 1$. Then, at

[31] This follows by an argument in our existence proof: if $\hat{q}_t(s^t) = 0$ for some (t, s^t), consumptions would become unbounded as $\ell \to \infty$, contradicting market clearing.

any node (t, s^t), $x \in X_{it}(s^t)$ if and only if

$$\sum_{s=1}^{S} Q_{it+1}(s^t, s)x(s) \geq \sum_{s=1}^{S} Q_{jt+1}(s^t, s)x(s) \text{ for all } j$$
$$\Leftrightarrow \sum_{s=1}^{S-1} a_{it+1}(s^t, s)x(s) + b_{it}(s^t) \geq \sum_{s=1}^{S-1} a_{it+1}(s^t, s)x(s) + b_{it}(s^t) \text{ for all } j,$$

where $a_{it+1}(s^t, s) \equiv Q_{it+1}(s^t, s) - Q_{it+1}(s^t, S)$ and $b_{it}(s^t) \equiv Q_{it+1}(s^t, S)$. Therefore, the optimal payoff sets can be viewed as the intersection of a $S-1$-dimensional power-diagrams with the $S-1$-dimensional simplex.

Any $S-1$ dimensional power diagrams maps into a collection of optimal payoff sets. The converse is also true. Given any $S-1$ dimensional power diagram, then there exists an economy such that the collection of optimal payoff sets correspond to the intersection of this power-diagram with the $S-1$ simplex. Namely, consider, at each node (t, s^t), $t < T$, a $S-1$ power diagram described by coefficients $(a_{t+1}(s^t), b_{t+1}(s^t))$ as above. Let

$$\Psi_{it+1}(s^t, s) \equiv a_{it+1}(s^t, s) + b_{it}(s^t) + K, \text{ for } s \in \{1, 2, \ldots, S-1\}$$

$$\Psi_{it+1}(s^t, S) \equiv b_{it}(s^t) + K,$$

where K is some constant that is large enough so that all $\Psi_{it+1}(s^t, s) > 0$ for all agents and all nodes. Then consider an economy with I agents, all endowed with identical CRRA preferences with parameter γ. Define the endowment of agent i recursively as

$$e_{it+1}(s^t, s) = g_{it+1}(s^t, s)e_{it}(s^t, s),$$

with initial condition $e_{i0}(s^0) = 1$, and where the growth rate is taken to solve:

$$\beta\pi_{t+1}(s \mid s^t)g_{it+1}(s^t, s)^{-\gamma} = \Psi_{it+1}(s^t, s).$$

If there is no collateral, $\bar{N} = 0$, it follows from the results in Section VIII that the private valuations $Q_{it+1}(s^t, s)$ for Arrow-Debreu securities take the form:

$$Q_{it+1}(s^t, s) = (1-\theta)\max_j \Psi_{jt+1}(s^t, s) + \theta\Psi_{it+1}(s^t, s).$$

But the first vector of the sum, $(1-\theta)\max_j \Psi_{jt+1}(s^t, s)$, is independent of i. It follows that $Q_{it+1}(s^t) \cdot x \geq Q_{jt+1}(s^t) \cdot x$ for all j if and only if $\Psi_{it+1}(s^t, s) \cdot x \geq \Psi_{jt+1}(s^t, s) \cdot x$ for all j, and so if and only if the intersection of the optimal payoff sets with the $S-1$ dimensional simplex belongs to power diagram we started with.

Assumptions and computations for Figure 3. Focus on two time periods t and $t+1$ in an economy with $I = 50$ agents with identical relative risk aversion $\gamma = 1$ and discount factor $\beta = 0.95$. There are three states, one disaster state $s = 1$ and two normal states, $s = 2$ and $s = 3$. State $s = 1$ is a disaster state with dismal endowment growth $g(3) = 0.8$, and realizes with low probability $\pi(3) = 0.05$. State $s = 2$ is a normal state with low endowment growth $g(2) = 1.01$, and realizes with probability $\pi(2) = 0.475$. State $s = 3$ is a normal state with high endowment growth $g(3) = 1.03$, and realizes with probability $\pi(3) = 0.475$. We assume that all agents have the same labor income at time t, normalized to $e_{it}(s^t) = 1$. At time $t+1$, on the other hand:

$$\frac{e_{it+1}(s^t, s)}{e_{it}(s^t)} = \alpha_i(s) g(s),$$

where the $\alpha_i(s)$ are drawn at random and are subsequently normalized so that $g(s)$ represents the aggregate growth of income in state s. We assume that there is no collateral, $\bar{N}(\Delta) = 0$, so that the intertemporal marginal rate of substitution of agent i is given by $\beta \pi_{t+1}(s \mid s^t) \left(\alpha_i(s) g(s)\right)^{-\gamma}$. In Appendix VIII we show that equilibria are continuous in $\bar{N}$ near $\bar{N}(\Delta) = 0$: hence, the optimal payoff sets with $\bar{N} = 0$ approximate those with $\bar{N} \simeq 0$.

Next, we reduce the dimension of the problem by studying the intersection of X_i with the simplex, as outlined above. Namely, proceeding as above, direct calculations reveals that this intersection is made up of points $x \in \mathbb{R}^2_+$, $x(1) + x(2) \leq 1$, such that

$$a_i \cdot x + b_i \geq a_j \cdot x + b_j \text{ for all } j,$$

where

$$\Psi_i(s) = \beta \pi(s) \left[\alpha_i(s) g(s)\right]^{-\gamma}$$

$$a_i(1) = \Psi_i(1) - \Psi_i(3)$$

$$a_i(2) = \Psi_i(2) - \Psi_i(3)$$

$$b_i = \Psi_i(3).$$

To calculate the optimal payoff sets, we use the `transport` package in R, which requires to reformulate the inequalities $a_i \cdot y + b_i \geq a_j \cdot y + b_j$ as

$$\|y - z_i\|^2 - w_i \leq \|y - z_j\|^2 - w_j,$$

for some $z_i \in \mathbb{R}^2$ and $w_i \geq 0$. Direct calculations show that $z_i = a_i/2$ and $w_i = \|z_i\|^2 + b_i$.

The calculation. Figure 3 is created using R, in the following steps. First, we use the fonction `power_diagram`, from the `transport` pacakge, to calculate the optimal payoff sets. Precisely, this function calculates the vertices of the polygons that enclose the intersection of optimal payoff sets with the unit square, $[0,1] \times [0,1]$. We then calculate the intersection of these polygons with the simplex, using the function `st_intersection` from the `sf` package. Finally, we plot these polygons in barycentric coordinates using the function `geom_polygon` of the `ggplot2` package.

X Parametric cases not covered in Proposition 3

Case $\gamma\phi = 1$. Then $V(\alpha, x)$ is linear in α and one can directly verify that the optimal payoff set of $\alpha_1 = 0$ is $X_0 = [0, x^\star(0)]$, where $x^\star(0)$ is defined in equation (52). The optimal payoff set for α_I is $X_I = [x^\star(0), 1]$. For all other agents $X_i = \{x^\star(0)\}$.

Case $\gamma\phi > 1$. In this case the function $\alpha \mapsto V(\alpha, x)$ is strictly convex in α. This implies that it achieves a strict maximum either at α_1 or at α_I. Therefore there exists some $x^\star$ such that $X_1 = [0, x^\star]$ and $X_I = [x^\star, 1]$. This threshold solves $V(\alpha_1, x^\star) = V(\alpha_I, x^\star)$, which leads to:

$$x^\star = \frac{\pi_2\left(1-(1-k_2\alpha_I)^{\phi\gamma}\right)}{\pi_1\left((1+k_1\alpha_I)^{\phi\gamma}-1\right)+\pi_2\left(1-(1-k_2\alpha_I)^{\phi\gamma}\right)}.$$

For all other i, $X_i = \emptyset$.

X.1 Proof that the price function solves (31)

The proof follows a standard-optimality verification argument. Let $\hat{P}_t(\delta\, s^t)$ denotes the solution of the Bellman equation (26) for tree δ, and let $\hat{J}$ denote a sequence of agent that solve the Bellman equation for tree δ at each node. Now take any stream of asset holders J. The Bellman equation (31) and the definition of $q_t(s^t \mid J)$ imply that at node (u, s^u), $u < T$:

$$\hat{P}_u(\delta \mid s^u) \geq \sum_s \frac{q_{u+1}(s^u \mid J)}{q_u(s^u \mid J)}\left[\delta_{u+1}(s^u, s) + \hat{P}_{u+1}(\delta \mid s^u, s)\right],$$

with an equality if j is the term (t, s^t) of the sequence of agent $\hat{J}$ solving the Bellman equation. Multiplying through by $q_u(s^u \mid J)$, dividing by $q_t(s^t \mid J)$ and adding the inequalities for all $s^u \succeq s^t$, we obtain that

$$\hat{P}_t(\delta \mid s^t) \geq \sum_{(u,s^u)\succ(t,s^t)} \frac{q_u(s^u \mid J)}{q_t(s^t \mid J)}\delta_u(s^u, s),$$

with an equality if $J = \hat{J}$. Therefore $\hat{P}(\delta \mid s^t)$ is an upper bound for $P(\delta \mid s^t)$, and it is attained for $\hat{J}$. The result follows.

XI A micro-foundation à la Rampini and Viswanathan (2010)

In this section we offer an alternative micro-foundation of the incentive constraint (6). We closely follow the approach of Rampini and Viswanathan (2010) and adapt their argument to our setting.

Precisely, let us imagine a contracting problem between an agent of our model and a continuum of competitive lenders. Agents and lenders take as given some no-arbitrage security prices $(p, q) \in \mathbb{NA}$, i.e. the prices of trees p and of Arrow securities q. As everywhere else in this Appendix, (p, q) denote time-zero deflated prices. Lenders can commit to make state-contingent payments to the agent, but they cannot hold trees: they cannot operate the technology that produces the corresponding dividend streams. The agent, on the other hand, can operate the technology but she has limited commitment. At any node (t, s^t), the agent can default, divert a fraction θ of the cum-dividend value of her tree portfolio, contract with a new competitive lender and continue to receive her stream of labor income. Agents maximize the same intertemporal utility as in the paper, while lenders maximizes the present value of their profits, evaluated at Arrow security prices.

XI.1 The optimal contracting problem

A contract is a list (c, N, τ) that specifies, for all nodes (t, s^t), the consumption and tree holdings of the agent, $c_t(s^t)$ and $N_t(s^t)$, as well as the net transfers made by the lender to the agent, $\tau_t(s^t)$. We omit the type "i" subscript for notational simplicity. A contract is feasible if it satisfies the budget, participation, and enforcement constraints specified below.

The key difficulty is to formulate the enforcement constraint at node (t, s^t), because it depends on the value of defaulting multiple times with successive lenders. Specifically, the enforcement constraint is determined by the value of defaulting and re-contracting with a new lender, which itself depends on the option to defaulting and re-contracting later with yet another lender, and so on. This implies that the enforcement constraint at node (t, s^t) depends on the enforcement constraint at successor nodes, $(u, s^u) \succ (t, s^t)$. Hence, it is natural to define the enforcement constraint recursively, by backward induction.

Namely, we first define the set of feasible contract at a terminal node, $B_T(w \mid s^T)$, where w is the agent's wealth, to be the collection of contracts (c, N, τ) such that

$$
\begin{aligned}
&q_T(s^T)c_T(s^T) = q_T(s^T)w + q_T(s^T)e_T(s^T) + q_T(s^T)\tau_T(s^T) \\
&q_T(s^T)\tau_T(s^T) = 0,
\end{aligned}
$$

i.e., the contracts that satisfy the budget constraint and the binding participation constraint of the lender, which we state as an equality for simplicity. Note that, at the terminal node, there are no trees left to purchase. As noted above, all valuations are based on time zero prices (p, q). We do not include an enforcement constraints at time T in $B_T(w \mid s^T)$: instead, we include these enforcement constraints at $T-1$, as explained below.

Now, proceeding by backward induction, consider any node (t, s^t), with $t < T$ and suppose that we have already constructed sets of feasible contracts, $B_u(w \mid s^u)$, for all successor nodes $(u, s^u) \succ (t, s^t)$. At node (t, s^t), we define $B_t(w \mid s^t)$ to be the set of contracts (c, N, τ) satisfying the following constraints. First, a budget constraint at (t, s^t):

$$
q_t(s^t)c_t(s^t) + \int p_t(\delta \mid s^t)\, dN_t(\delta\, s^t) = q_t(s^t)w + q_t(s^t)e_t(s^t) + q_t(s^t)\tau_t(s^t). \tag{75}
$$

Second a budget constraint at all $(u, s^u) \succ (t, s^t)$,

$$
\begin{aligned}
&q_u(s^u)c_u(s^u) + \int p_u(\delta \mid s^u)\, dN_u(\delta\, s^u) \\
&= \int \left[q_u(s^u)\delta_u(s^u) + p_u(\delta \mid s^u)\right]\, dN_{u-1}(\delta \mid s^{u-1}) + q_u(s^u)e_u(s^u) + q_u(s^u)\tau_u(s^u).
\end{aligned} \tag{76}
$$

Third, the binding participation constraint for the lender at (t, s^t):

$$
\sum_{(u,s^u)\succeq(t,s^t)} q_u(s^u)\tau_u(s^u) = 0. \tag{77}
$$

Fourth, a set of enforcement constraints for the agent, that ensures that, at all successor nodes $(u, s^u) \succ (t, s^t)$, the agent does not have incentive to default:

$$
\sum_{(v,s^v)\succ(u,s^u)} \beta^v \pi_v(s^v)u(c_v(s^v)) \geq \sum_{(v,s^v)\succ(u,s^u)} \beta^v \pi_v(s^v)u(\hat{c}_v(s^v)), \tag{78}
$$

for all $(\hat{c}, \hat{N}, \hat{\tau}) \in B_u(\hat{w}_u(s^u) \mid s^u)$, where

$$
q_u(s^u)\hat{w}_u(s^u) = \theta \int \left[q_u(s^u)\delta_u(s^u) + p_u(\delta \mid s^u)\right]\, dN_{u-1}(\delta \mid s^{u-1}),
$$

is the value of the agent's tree portfolio after default and diversion. This last constraint states formally that the agent can default, divert a fraction θ of the cum-dividend value of its portfolio and pick a feasible contract with a new lender. This new contract gives the agent the option to default at some future date. In particular, as anticipated above, the enforcement constraints at node (t, s^t) depends on future enforcement constraints encoded in the sets $B_u(\hat{w}_u(s^u) \,|\, s^u)$.

The *optimal contracting problem* is to choose $(c, N, \tau) \in B_0(w_0 \,|\, s^0)$, where w_0 is the cum-dividend value of the agent's initial tree portfolio endowment.

XI.2 Implementation with Arrow borrowing and incentive constraints

The main result is that the agent's problem considered in the text, with incentive constraints of the form (6), yields the same allocation as the optimal contracting problem defined above. Formally:

Proposition XI.1 *Consider a solution (c, N, a) to the agent's problem considered in the text, and let*

$$q_t(s^t)\tau_t(s^t) \equiv q_t(s^t)a_t(s^t) - \sum_s q_{t+1}(s^t, s)a_{t+1}(s^t, s).$$

Then (c, N, τ) solves the optimal contracting problem.

The proof proceeds in two steps.

Step 1 of the proof. Similarly to Rampini and Viswanathan (2010) we show that the enforcement constraint implies:

Lemma XI.1 *If $(c, N, \tau) \in B_0(w_0 \,|\, s^0)$ then, at all nodes (t, s^t), $t > 0$:*

$$-\sum_{(u,s^u)\succeq(t,s^t)} q_u(s^u)\tau_u(s^u) \leq (1-\theta)\int \left[q_t(s^t)\delta_t(s^t) + p_t(\delta \,|\, s^t)\right]\, dN_{t-1}(\delta \,|\, s^{t-1}). \tag{79}$$

According to (79), the present value of net transfers to the lender cannot must be less than $1 - \theta$ times the cum-dividend value of the tree portfolio acquired in the previous period. The proof, which is identical to that in Rampini and Viswanathan (2010), proceeds by contradiction. Suppose that (79) does not hold at some node (t, s^t). Then, we argue that the incentive constraint (78) is violated at this node. Indeed, the agent could default, start over with $\hat{w}_t(s^t) = \theta \int \left[q_t(s^t)\,\delta_t(s^t) + p_t(\delta \,|\, s^t)\right]\, dN_{t-1}(\delta \,|\, s^{t-1})$ at node (t, s^t) and pick a new contract $(\hat{c}, \hat{N}, \hat{\tau}) \in B_t(\hat{w}_t(s^t) \,|\, s^t)$ that changes the transfer and consumption at node (t, s^t), but keeps everything else the same. Namely, the transfer at (t, s^t) is changed to:

$$q_t(s^t)\hat{\tau}(s^t) = -\sum_{(u,s^u)\succ(t,s^t)} q_t(s^t)\tau_t(s^t),$$

while consumption is changed to:

$$\begin{aligned} & q_t(s^t)\left[\hat{c}_t(s^t) - c_t(s^t)\right] \\ = & -(1-\theta)\int \left[q_t(s^t)\delta_t(s^t) + p_t(\delta \,|\, s^t)\right]\, dN_{t-1}(\delta \,|\, s^{t-1}) - \sum_{(u,s^u)\succeq(t,s^t)} q_u(s^u)\tau_u(s^u) \end{aligned}$$

which is strictly positive by our maintained assumption. The budget constraints and the participation constraint of the lenders hold by construction. Moreover, since the original contract satisfies the enforcement constraint at all nodes $(u, s^u) \succ (t, s^t)$, so does the new contract. But this means that the enforcement constraint does not hold at (t, s^t), a contradiction.

Step 2 of the proof. With the Lemma in mind, we define the *relaxed problem* to be the optimal contracting problem in which all enforcement constraints are replaced by (79). Clearly the Lemma shows that the constraint set in the original problem is contained in that of the relaxed problem. Hence, the value of the relaxed problem is an upper bound to the value of the original problem. Next, we show that this upper bound is attained:

Lemma XI.2 *The value of the original contracting problem is equal to that of the relaxed contracting problem, and to the agent's problem considered in the text.*

To establish this result we make the same change of variable as in Rampini and Viswanathan (2010), namely we let:

$$q_t(s^t)a_t(s^t) \equiv \sum_{(u,s^u)\succeq(t,s^t)} q_u(s^u)\tau_u(s^u)$$

$$\Leftrightarrow \quad q_t(s^t)\tau_t(s^t) = q_t(s^t)a_t(s^t) - \sum_s q_{t+1}(s^t, s)a_{t+1}(s^t, s),$$

with $a_0(s^0) = 0$. This leads to an equivalent representation of the constraint set of the relaxed problem. Namely, a contract (c, N, a) belongs to the constraint set of the relaxed problem starting at time t given initial financial wealth w if it satisfies the following constraints. First, a budget constraint at (t, s^t):

$$q_t(s^t)c_t(s^t) + \int p_t(\delta \,|\, s^t)\, dN_t(\delta\, s^t) + \sum_s q_{t+1}(s^t, s)a_{t+1}(s^t, s) = q_t(s^t)w + q_t(s^t)e_t(s^t). \tag{80}$$

Second a budget constraint at all $(u, s^u) \succ (t, s^t)$,

$$q_u(s^u)c_u(s^u) + \int p_u(\delta \,|\, s^u)\, dN_u(\delta \,|\, s^u) + \sum_s q_{u+1}(s^u, s)a_{u+1}(s^u, s) \tag{81}$$

$$= \int \left[q_u(s^u)\delta_u(s^u) + p_u(\delta \,|\, s^u)\right]\, dN_{u-1}(\delta \,|\, s^{u-1}) + q_u(s^u)e_u(s^u) + q_u(s^u)a_u(s^u). \tag{82}$$

with the convention that $N_T = a_{T+1} = 0$. Third a set of incentive constraints for for all $(u, s^u) \succ (t, s^t)$, which take the same form as (6):

$$-q_u(s^u)a_u(s^u) \leq (1-\theta)\int \left[q_u(s^u)\delta_u(s^u) + p_u(\delta \,|\, s^u)\right]\, dN_{u-1}(\delta \,|\, s^{u-1}). \tag{83}$$

Clearly the relaxed optimal contracting problem at time $t = 0$ with wealth w_0 has become identical to the agent's problem studied in the text.

Now let $V_t(w \,|\, s^t)$ denote the value of the relaxed contracting problem starting with financial wealth w at (t, s^t). Clearly, this value function is weakly increasing in w. Moreover a standard dynamic programming argument shows that if (c, N, a) solves the relaxed contracting problem starting at time zero given w_0, then it also solves it starting at node (t, s^t) given financial wealth

$$q_t(s^t)w_t(s^t) = \int \left[q_t(s^t)\delta_t(s^t) + p_t(\delta \mid s^t)\right] \, dN_{t-1}(\delta \mid s^{t-1}) + a_t(s^t).$$

We now verify that a solution of the relaxed contracting problem is incentive compatible for the original contracting problem (after doing the opposite change of variables from a to τ). Indeed, the budget constraints and participation constraint hold by construction. The enforcement constraint (78) hold as well at all (t, s^t), $t > 0$. Indeed, the left-hand side of the enforcement constraint is equal to $V_t(w_t(s^t) \mid s^t)$ defined above. The right-hand side is bounded above by the solution of the relaxed contracting problem, starting with financial wealth

$$q_t(s^t)\hat{w}_t(s^t) = \theta \int \left[q_t(s^t)\delta_t(s^t) + p_t(\delta \mid s^t)\right] \, dN_{t-1}(\delta \mid s^{t-1}).$$

Since the incentive constraint (83) holds at (t, s^t), we have that $w_t(s^t) \geq \hat{w}_t(s^t)$. Since the value function is weakly increasing, $V_t(w_t(s^t) \mid s^t) \geq V_t(\hat{w}_t(s^t) \mid s^t)$. Therefore, the enforcement constraint holds.