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Incentive Constrained Risk Sharing, Segmentation, and Asset Pricing†

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Incentive problems make securities' payoffs imperfectly pledgeable, limiting agents' ability to issue liabilities. We analyze the equilibrium consequences of such endogenous incompleteness in a dynamic exchange economy. Because markets are endogenously incomplete, agents have different intertemporal marginal rates of substitution, so that they value assets differently. Consequently, agents hold different portfolios. This leads to endogenous markets segmentation, which we characterize with optimal transport methods. Moreover, there is a basis going always in the same direction: the price of a security is lower than that of replicating portfolios of long positions. Finally, equilibrium expected returns are concave in factor loadings. (*JEL* D51, D52, G11, G12)

One of the key functions of financial markets is to enable agents to share risk. For example, relatively risk tolerant market participants can sell puts or credit default swaps to more risk averse agents, or agents with larger risk exposure. We study risk sharing in general equilibrium in a dynamic exchange economy. At each period agents receive heterogeneous endowments of consumption good (labor income), as well as the output or "dividends" of the assets, or "trees," they hold. At each period there is also a complete set of zero net supply Arrow securities, spanning endowments and dividends.

Agents' liabilities, modeled as sales of Arrow securities, are backed by the assets they can pledge as collateral. Should these participants default, however,

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seizing their collateral could prove difficult and costly. This has been documented for a variety of collateral assets, for example residential homes backing mortgages (Campbell, Giglio, and Pathak 2011), productive assets backing firms' liabilities (Andrade and Kaplan 1998), and traded assets backing financial firms' liabilities (Fleming and Sarkar 2014).

That seizing collateral is difficult and costly creates scope for opportunistic debtors' behavior. In Kiyotaki and Moore (1997), strategic debtors use the threat of costly bankruptcy to negotiate debt down to liquidation values. We consider a similar mechanism. Suppose an agent issued Arrow securities, promising to pay a given amount should a given state occur. When that state realizes, the agent can threaten to default on her promise. Suppose that, in case of default, the buyer of the security can only seize a fraction $1 - \theta$ of the assets of the defaulting agent, while the fraction θ is deadweight bankruptcy cost. In this context, if the agent can make a take-it-or-leave-it offer to the Arrow security buyer, she can renegotiate her liability to a fraction $1 - \theta$ of the value of her asset holdings. We refer to this value as the pledgeable income of the agent, and to the constraint that the agent cannot promise more than her pledgeable income as the incentive constraint. Imperfect pledgeability implies over-collateralization and limits agents' ability to sell Arrow securities, which generates endogenous market incompleteness. The goal of this paper is to study the consequences of incentive constraints and imperfect collateral pledgeability for risk sharing, portfolio choice, and asset and derivative pricing.

Because, in addition to classical budget constraints, agents face incentive constraints in which prices enter, standard equilibrium existence proofs based on welfare theorems do not apply in our framework. In particular, the approach pioneered by Negishi (1960) cannot be used. Yet, we prove equilibrium existence, extending the price-player proof of Arrow and Debreu (1954) to our setting.

In a frictionless complete market equilibrium, intertemporal marginal rates of substitution are equalized across agents. This yields a valuation operator (or pricing kernel), common to all agents, which prices all securities. In contrast, when incentive constraints limit risk sharing, agents have different intertemporal marginal rates of substitution and thus have different private valuations for imperfectly pledgeable assets. In equilibrium, each tree is held by the agent who values it most. Because agents value trees differently, they hold strictly different portfolios; i.e., there is endogenous segmentation. Intuitively, agents choose tree portfolios that, in combination with their labor income, come close to replicate their desired consumption profile. Then, to further approach their desired consumption, agents either buy Arrow securities, or they use their tree portfolios as collateral to sell Arrow securities. Theoretically, we show that the equilibrium allocation of trees to agents solves the classical optimal transport problem of drawing power diagrams (Galichon 2016, chap. 5). Empirically, equilibrium segmentation is in line with evidence from household finance. For example, Catherine, Sodini, and Zhang (2020) find that "workers facing higher left-tail income risk when equity markets perform poorly are less likely to participate in the stock market."

In our framework as in previous models of endogenously incomplete markets (notably Alvarez and Jermann 2000), only agents whose incentive constraints do not bind in a given state have intertemporal marginal rates of substitution, i.e., private valuations, equal to the Arrow security price of that state. And it is those agents who in equilibrium buy these Arrow securities. In contrast, the other agents' intertemporal marginal rates of substitution for that state are lower than the Arrow security price. They sell Arrow securities until their incentive constraint binds.

Therefore, as soon as an agent's incentive constraint binds in at least one state, this agent's private valuation for a tree paying off in that state is lower than the price of a replicating portfolio of Arrow securities. When this is true for all agents, the tree is priced below its replicating portfolio; i.e., there is a basis. More generally, any asset is priced below any portfolio of long positions in assets or securities that replicates its payoff. Such deviations from the law of one price are equilibrium phenomena, which cannot be arbitraged. To engage in arbitrage, one would have to sell the expensive leg of the arbitrage, i.e., sell Arrow securities. Such sales, however, would violate incentive constraints.

A tree can be viewed as a bundle of imperfectly pledgeable payoffs in different states. It is priced below any replicating portfolio of long positions in assets and securities because each asset in the replicating portfolio can be held by the agent who values it most, whereas the tree must be held by a unique agent. The inequality is strict when there is no agent who has the highest private valuation for all the assets in the replicating portfolio.

For example, the payoff of a convertible bond is identical to the payoff of a portfolio combining a straight bond and a call option. In line with empirical evidence (Mitchell and Pulvino 2012), our model implies that in equilibrium, convertible bonds can be priced strictly below the replicating portfolio of straight bond and call. In practice, to take advantage of that arbitrage opportunity, market participants such as hedge funds buy convertibles and issue straight bonds and calls. Such arbitrage is constrained, however, both in practice as in our model, by the limited ability of hedge funds to issue the replicating securities.

The observation that trees are less valuable than replicating portfolios suggests that equilibrium outcomes are not invariant to changes in the tree supply, holding aggregate tree dividend and everything else the same. In particular, breaking up trees into replicating portfolios changes the tree supply in a way that relaxes incentive constraints and improves risk sharing. In contrast, when trees are fully pledgeable, the manner in which aggregate dividends are split across trees is irrelevant.

The basis between trees and replicating portfolios has a prediction for the cross section of expected returns. Project the returns of the trees on a set of factors. If the residuals of this projection are orthogonal to the agents' private valuations (which is the case, in particular, when the factors are themselves the agents' private valuation), then expected returns are concave in factor loadings. Our theoretical result that equilibrium returns are concave in factor loadings, i.e., betas, is consistent with the empirical finding that the security market line is concave (Frazzini and Pedersen 2014, Hong and Sraer 2016).

Related Literature

Our analysis of dynamic general equilibrium and endogenous incompleteness is in line with the seminal analyses of Kehoe and Levine (1993), Alvarez and Jermann

(2000), Chien and Lustig (2010), and Gottardi and Kubler (2015).[1] The main difference between our model and theirs is that we consider assets that are, at the same time, imperfectly pledgeable and tradable. Thus, we import in a limited enforcement asset pricing model similar to Chien and Lustig (2009), the assumption that assets are imperfectly pledgeable, as in Rampini and Viswanathan (2010). The main difference between our analysis and Rampini and Viswanathan (2010) is that they analyze a production economy with investment, but take asset prices as exogenously given, while we consider an exchange economy, but endogenize prices. The main difference between our results and Kehoe and Levine (1993), Alvarez and Jermann (2000), Chien and Lustig (2009), and Gottardi and Kubler (2015) is that we obtain equilibrium deviations from the law of one price and endogenous segmentation.

Our result that market imperfections lead to deviations from the law of one price is in line with Hindy and Huang (1995), Gromb and Vayanos (2002, 2018), Gârleanu and Pedersen (2011). One difference is that we provide a micro-foundation for financial constraints in terms of imperfect collateral pledgeability. This leads to our new result that markets are endogenously segmented. In Gromb and Vayanos (2002), in contrast, segmentation is exogenous. Also, while Gârleanu and Pedersen (2011) study bases among assets and securities with different exogenous margin constraints, in our model all assets and securities have identical pledgeability, yet bases for different assets are endogenously different.

Geanakoplos and Zame (2014), Fostel and Geanakoplos (2008), Brumm et al. (2015), Geerolf (2015), and Lenel (2017) also analyze general equilibrium under collateral constraints. In that literature, each financial promise must be backed by its own collateral, which gives rise to over-collateralization as shown by Araújo, Kubler, and Schommer (2010).[2] In our framework, by contrast, the constraint applies to the portfolio of assets and Arrow securities of an agent, in line with the practice of portfolio margining.[3] Yet, imperfect pledgeability generates over-collateralization.

In our model pledgeable payoffs are discounted less than non-pledgeable ones. This is in line with the collateral premium analyzed by Geanakoplos and Zame (2014), Fostel and Geanakoplos (2008), and the liquidity premium derived by the new monetarist literature (see, for example, Lagos 2010; Li, Rocheteau, and Weill 2012; Lester, Postlewaite, and Wright 2012; Venkateswaran and Wright 2013; Jacquet 2015). Moreover, while the pricing of pledgeable payoffs is linear and based on a single stochastic discount factor, the pricing of non-pledgeable payoffs is nonlinear and convex, based on multiple stochastic discount factors. This implies that equally pledgeable payoffs are priced differently depending on their state-contingent structure, leading to bases between assets and replicating portfolios, and to concave factor pricing.

Methodologically, our paper shows that the incentive-constrained allocation of assets across agents can be characterized with techniques from optimal transport

[1] Lustig and Van Nieuwerburgh (2010) analyze empirical implications of this framework.

[2] For example, the same asset generating strictly positive output in two states cannot be used to collateralize the issuance of two Arrow securities, promising payments in these two states.

[3] For example, on http://www.cboe.com/products/portfolio-margining-rules, one can read, "The portfolio margining rules have the effect of aligning the amount of margin money … to the risk of the portfolio as a whole, calculated through simulating market moves up and down, and accounting for offsets between and among all products held …."

theory. This means that the problem of pricing and allocating assets (bundles of risk) to heterogeneous agents is economically similar to that of compensating and assigning workers (bundles of skills) to heterogeneous firms. See Rosen (1983), Heckman and Scheinkman (1987), and, more recently, Edmond and Mongey (2019). We contribute to the analysis of this problem by considering state-contingent borrowing, in effect an imperfect technology to unbundle risks, and by making the assignment problem dynamic. Another difference is that, in our setting, welfare theorems do not hold, so existence cannot be established via optimization.

The remainder of this paper has two parts: Section I describes the model and Section II analyzes the equilibrium. The main and secondary proofs are in the print and online Appendices.

I. Model

A. *Assets and Agents*

There is a finite number of time periods $t \in \{0, 1, \ldots, T\}$. Every period a new state is drawn from some finite set S. We let $s_t \in S$ denote the state in period t, $s^t = (s^{t-1}, s_t)$ the history of states until t, and S^t the set of time-t histories starting from s_0. The probability of history s^t, conditional on s_0, is denoted by $\pi_t(s^t)$ and is assumed to be strictly positive. A node in the event tree is a pair (t, s^t) of time $t \leq T$ and history $s^t \in S^t$.

At every node (t, s^t), $t < T$, there is a complete set of one-step-ahead Arrow securities in zero net supply. In addition to Arrow securities, there are trees in positive supply. A tree is defined by its dividend stream $\delta \equiv \{\delta_t(s^t) : t \geq 0, s^t \in S^t\}$, i.e., the collection of its dividend payouts for all nodes. We do not impose any restriction on the set Δ of dividend streams except that $\delta_t(s^t) \in [0, \bar{\delta}]$.[4] Figure 1 illustrates. For example, the set Δ can contain short-lived trees with payoffs identical to Arrow securities, long-lived trees, bonds of arbitrary maturity, and so on.

We represent the supply of trees by some positive and finite measure $\bar{N}$ over the set Δ, endowed with its Borel σ-algebra. A special case is the standard model with a finite number of trees in positive supply. But our results apply equally to arbitrary supplies over Δ, defined by continuous measures, discrete measures, or mixtures of both. This added generality serves several purposes. First, it clarifies the analysis of market segmentation, by providing simple geometrical representations for the equilibrium allocation trees, and establishing connections with classical results in optimal transport theory. Second, it demonstrates that our results are not driven by some form of market incompleteness.[5]

[4] The upper bound $\bar{\delta}$ is arbitrary and can be viewed as a normalization, since agents can always increase the dividend payout of a tree proportionally at all nodes by scaling up their holdings. Technically, the upper bound facilitates the analysis because it makes both the set of trees, Δ, and bounded sets of positive measure over Δ, compact in appropriate topologies (for the latter, see chapter 15 in Aliprantis and Border 2006).

[5] In particular, while we focus on one-step-ahead Arrow securities, for any multiple-step-ahead Arrow security, there exists a tree that has exactly the same payoff. While for simplicity for the moment we rule out short positions in trees, in Corollary 1 we show that the equilibrium we characterize remains an equilibrium when agents can short trees. In this sense, our focus on one-period-ahead Arrow securities is without loss of generality.

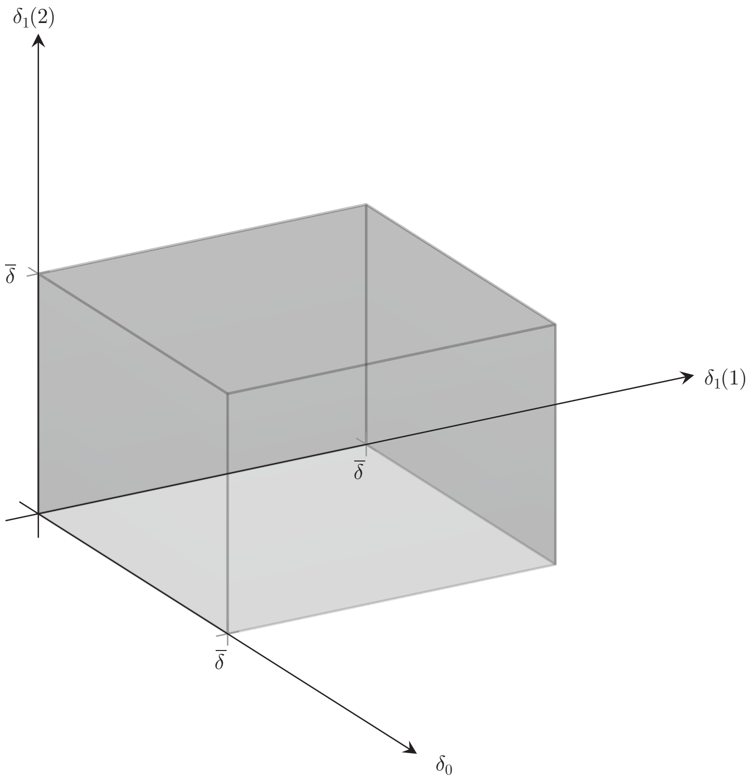

Figure 1. The Set Δ When There Are Two Periods, $t \in \{0,1\}$, and Two States, $S = \{1,2\}$

To facilitate the proof of equilibrium existence, we assume that the distribution of tree supplies is such that the aggregate dividend is strictly positive at all nodes, that is,

$$\int \delta_t(s^t)\, d\bar{N}(\delta) > 0, \tag{1}$$

for all (t, s^t), where the integral is taken over Δ and $d\bar{N}(\delta)$ is the supply of trees with dividend streams δ.[6]

On the other side of the market there is a finite number of agent's types indexed by i, with a measure one of each, who order consumption plans $c_i \equiv \{c_{it}(s^t), 0 \le t \le T, s^t \in S^t\}$ according to the intertemporal utility:

$$\sum_{(t,s^t)} \beta^t \pi_t(s^t)\, u_i(c_{it}(s^t)), \tag{2}$$

where $u_i(c)$ is strictly increasing, concave, continuous, and continuously differentiable over $c > 0$. We also assume continuity at $c = 0$ unless $u_i(c)$ is unbounded below, for example in the case of log utility. Agent i starts at time zero with no

[6] If there is a finite number of trees, the measure-theoretic notation $d\bar{N}(\delta)$ can be replaced with $\bar{n}(\delta)$, the mass of trees with dividend stream δ, and equation (1) writes as $\sum_{\delta \in \Delta} \delta_t(s^t)\bar{n}(\delta) > 0$.

endowment of Arrow securities and with a tree endowment equal to a fraction $\alpha_i > 0$ of the market portfolio, $N_{i,-1} \equiv \alpha_i \bar{N}$ where $\sum_i \alpha_i = 1$. The agent also receives at every node (t, s^t), an endowment of $e_{it}(s^t) \geq 0$ consumption good which we will refer to as labor income.

B. *Agents' Budget and Incentive Constraints*

With Arrow Securities.—At each node (t, s^t), $t < T$, agent i consumes $c_{it}(s^t) \geq 0$, takes long tree positions represented by a positive and finite measure over Δ, denoted by $N_{it}(s^t)$, and takes net positions (long minus short) $a_{it+1}(s^t, s)$ in Arrow securities paying off in state s, for all $s \in S$. We show later, in Corollary 1, that the short-selling constraint for trees is not binding. Letting $P_t(\delta | s^t)$ denote the continuous price function for trees and $Q_{t+1}(s^t, s)$ the price of Arrow securities at node (t, s^t), the sequential budget constraint for $t < T$ writes

$$c_{it}(s^t) + \int P_t(\delta | s^t)\, dN_{it}(\delta | s^t) + \sum_s Q_{t+1}(s^t, s)\, a_{it+1}(s^t, s) = e_{it}(s^t) + \int \left[\delta_t(s^t) + P_t(\delta | s^t)\right] dN_{it-1}(\delta | s^{t-1}) + a_{it}(s^t), \tag{3}$$

where $a_{i0}(s^0) = 0$ and $N_{i,-1} = \alpha_i \bar{N}$ and $dN_{it}(\delta | s^t)$ denotes the number of trees with dividend stream δ purchased by agent i at node (t, s^t). At $t = T$, the constraint writes

$$c_{iT}(s^T) = e_{iT}(s^T) + \int \delta_T(s^T)\, dN_{iT-1}(\delta | s^{T-1}) + a_{iT}(s^T). \tag{4}$$

We assume that, at each node starting at $t = 1$, an agent can threaten to default, in which case its creditors obtain fraction $1 - \theta$ of all long positions, for some $\theta \in (0,1)$.[7] If the agent can make a take-it-or-leave-it offer to its creditors, the maximum amount it can credibly promise when selling Arrow securities is given by

$$a^-_{it+1}(s^t, s) \leq (1 - \theta)\left\{ a^+_{it+1}(s^t, s) + \int \left[\delta_{t+1}(s^t, s) + P_{t+1}(\delta | s^t, s)\right] dN_{it}(\delta | s^t) \right\}, \tag{5}$$

for all (t, s^t), $t < T$, and s, where $a^-_{it+1}(s^t, s) = \max\{-a_{it+1}(s^t, s), 0\}$ is the short position in Arrow security, and $a^+_{it+1}(s^t, s) = \max\{a_{it+1}(s^t, s), 0\}$ is the long position. Thus, we assume two-sided limited commitment in that the agent who sold Arrow securities cannot commit not to renegotiate her liabilities and the agent who bought Arrow securities cannot commit to reject take-it-or-leave-it offers.[8]

Since (5) always holds if $a_{it+1}(s^t, s) \geq 0$, it can be simplified into

$$-a_{it+1}(s^t, s) \leq (1 - \theta) \int \left[\delta_{t+1}(s^t, s) + P_{t+1}(\delta | s^t, s)\right] dN_{it}(\delta | s^t). \tag{6}$$

In other words, an agent's liability cannot be larger than a fraction $1 - \theta$ of its tree portfolio, the maximum amount it would repay given that it can threaten to default.

[7] In the Appendix, we study the agent's problem and prove equilibrium existence in a more general case: we assume that the parameter θ is both agent and tree specific.

[8] One may wonder what happens to the fraction θ of long position that is not recovered by creditors: is it diverted by the agent or is it a deadweight loss, for example a bankruptcy cost? It turns out that it does not matter for the argument because no default occurs in equilibrium.

We will refer to equation (6) as the *incentive constraint*, and to the right-hand side of (6) as the agent's *pledgeable income*.

The incentive constraint (6) generalizes that of Chien and Lustig (2009) by allowing collateral to be imperfectly pledgeable: we assume that $\theta > 0$ while Chien and Lustig assumed that $\theta = 0$. While we have derived the incentive constraint (6) based on ex post renegotiation, online Appendix XI offers an alternative micro-foundation based on limited enforcement and cash diversion, in line with Rampini and Viswanathan (2010).

With Cash on Hand.—Some of the analysis can be simplified with the following change of variable:

$$W_{it}(s^t) \equiv e_{it}(s^t) + \int \left[\delta_t(s^t) + P_t(\delta|s^t)\right] dN_{it-1}(\delta|s^{t-1}) + a_{it}(s^t).$$

In words, $W_{it}(s^t)$ represents the agent's cash-on-hand: the combined value of the endowment, the tree portfolio, and the Arrow security payoff. One advantage of the cash-on-hand formulation is to simplify notations by suppressing any explicit reference to Arrow securities. Indeed, with cash-on-hand, the sequential budget constraint becomes

$$(7) \quad c_{it}(s^t) + \int P_t(\delta|s^t)\, dN_{it}(\delta|s^t) + \sum_s Q_{t+1}(s^t,s)\, W_{it+1}(s^t,s)$$
$$= W_{it}(s^t) + \sum_s Q_{t+1}(s^t,s)\, e_{it+1}(s^t,s)$$
$$+ \sum_s Q_{t+1}(s^t,s) \int \left[\delta_{t+1}(s^t,s) + P_{t+1}(\delta|s^t,s)\right] dN_{it}(\delta|s^t),$$

for all (t,s^t), and with the convention that time $T+1$ variables and time T tree prices are equal to zero. Likewise, the incentive constraint (6) can be written

$$(8) \qquad W_{it+1}(s^t,s) \geq e_{it+1}(s^t,s) + \theta \int \left[\delta_{t+1}(s^t,s) + P_{t+1}(\delta|s^t,s)\right] dN_{it}(\delta|s^t),$$

for all (t,s^t), $t < T$, and s. Equation (8) states that agents' cash-on-hand in all successor nodes must be larger than the non-pledgeable income stemming from their labor endowment and tree payoff. This limits an agent's ability to hold trees whose payoff is high in the states in which she is constrained.

C. *Definition of Equilibrium*

A price system is some (P,Q), where $P \equiv \left\{P_t(\delta|s^t), 0 \leq t < T, s^t \in S^t\right\}$ is a sequence of positive and continuous price functions for trees, and $Q \equiv \left\{Q_{t+1}(s^t,s),\right.$ $\left. 0 \leq t < T, s^t \in S^t, s \in S\right\}$ is a sequence of Arrow security prices. Given (P,Q), an agent chooses plans for consumption, $c_i = \left\{c_{it}(s^t), 0 \leq t \leq T, s^t \in S^t\right\}$, and for tree portfolios, $N_i = \left\{N_{it}(s^t), 0 \leq t < T, s^t \in S^t\right\}$. A plan for consumption and tree portfolios, (c_i, N_i), is *budget feasible and incentive compatible* if there exists some plan for cash-on-hand $W_i = \left\{W_{it}(s^t), 0 \leq t \leq T, s^t \in S^t\right\}$ such that

(c_i, N_i, W_i) satisfies the budget constraint (7) at all nodes, the incentive constraint (8) at all nodes, and the initial condition:

$$W_{i0}(s^0) = e_{i0}(s^0) + \alpha_i \int \left[\delta_0(s^0) + P_0(\delta|s^0)\right] d\bar{N}(\delta). \tag{9}$$

The *agent's problem* is to choose a budget feasible and incentive compatible plan, (c_i, N_i), in order to maximize the intertemporal utility (2).

An allocation is a collection $(c_i, N_i)_{i \in I}$ of plans for consumption and tree portfolio. An allocation is *feasible* if, at all nodes (t, s^t)

$$\sum_i c_{it}(s^t) = \sum_i e_{it}(s^t) + \sum_i \int \delta_t(s^t)\, dN_{it-1}(\delta|s^{t-1}), \tag{10}$$

$$\sum_i N_{it}(s^t) = \bar{N}. \tag{11}$$

The feasibility condition for trees (11) states that the demand for dividend stream δ, $\sum_i dN_{it}(\delta|s^t)$, is equal to the supply, $d\bar{N}(\delta)$.

An *equilibrium* is a price system (P, Q) and a feasible allocation $(c_i, N_i)_{i \in I}$ such that, for all $i \in I$, (c_i, N_i) solves the problem of agent's i given (P, Q).

This definition is formulated in the spirit of a classical time-zero Arrow-Debreu equilibrium, in the sense that it suppresses any explicit reference to agents' positions in Arrow securities.[9] There is one important difference however, which sets our model apart from earlier work in the endogenous incomplete market literature, such as Alvarez and Jermann (2000), Chien and Lustig (2009), and Gottardi and Kubler (2015). In the Arrow-Debreu equilibria defined in these earlier works, agents do not explicitly trade trees: indeed no arbitrage implies that it is equivalent for agents to only trade claims to consumptions at all future nodes. Our definition, in contrast, must be explicit about agents' trades in trees. This is because trees are imperfectly pledgeable, implying, as shown below, that standard no-arbitrage relationships do not apply and trading trees is no longer equivalent to trading consumption claims.[10]

II. Equilibrium Analysis

A. *No-Arbitrage Relationships*

We first establish key no-arbitrage relationships that have to hold in our setting:

LEMMA 1: *Let (P, Q) and $(c_i, N_i)_{i \in I}$ be an equilibrium. Then,*

[9] See, for example, chapter 8 in Ljungqvist and Sargent (2012). It is routine to verify that the definition is equivalent to the corresponding one with Arrow securities. For example, using the sequential budget constraints (3), one can recover agents' implied Arrow securities positions, and verify that the market for Arrow securities clears.

[10] To be clear, Chien and Lustig (2009) and Gottardi and Kubler (2015) define Arrow-Debreu equilibria differently from us. In particular, they do not redefine incentive constraints based on a notion of cash-on-hand, but instead they show how to replace the collateral constraints by what they call "solvency constraints": namely at all nodes, the present value of consumption must be larger than that of the labor endowment. The key point is that, in Chien and Lustig (2009) and Gottardi and Kubler (2015), agents' tree portfolios do not enter these solvency constraints. We can derive solvency constraints in our setting as well, by iterating the incentive constraints (8) forward. But, in contrast to Chien and Lustig (2009) and Gottardi and Kubler (2015), imperfect pledgeability implies that these solvency constraints now depend on agents' tree portfolios.

(i) *The price of consumption is strictly positive at all* (t, s^t), $t < T$:

$$Q_{t+1}(s^t, s) > 0; \tag{12}$$

(ii) *Trees are priced at most at the value of their total payoff at all* (t, s^t), $t < T$:

$$P_t(\delta | s^t) \leq \sum_s Q_{t+1}(s^t, s)\left[\delta_{t+1}(s^t, s) + P_{t+1}(\delta | s^t, s)\right], \tag{13}$$

$\bar{N}$-almost everywhere in Δ.

(iii) *Trees are priced at least at the value of their pledgeable payoff:*

$$P_t(\delta | s^t) \geq (1-\theta)\sum_s Q_{t+1}(s^t, s)\left[\delta_{t+1}(s^t, s) + P_{t+1}(\delta | s^t, s)\right], \tag{14}$$

everywhere in Δ, with a strict inequality if the continuation dividend stream is nonzero.

For the first no-arbitrage relationship, suppose that the price of consumption were zero for some (t, s^t): then all agents would find it optimal to increase their consumption at that node, violating the market clearing condition for consumption.

For the second no-arbitrage relationship, suppose that at node (t, s^t), the price of some tree in positive supply were strictly larger than the present value of its total future payoffs. Then all agents holding this tree could sell it and purchase instead a replicating portfolio of Arrow securities, making a risk-free profit without violating their incentive constraint: indeed equation (8) shows that replacing a tree with a replicating portfolio of Arrow securities keeps cash-on-hand the same but reduces the non-pledgeable income stemming from the tree payoff. Hence, if the second no-arbitrage relationship did not hold, the market could not clear.[11]

Finally, for the third no-arbitrage relationship suppose that at some node (t, s^t), the price of some tree with nonzero continuation dividend stream were lower than the value of its pledgeable future payoffs. Then an agent could finance the purchase of the tree by selling a replicating portfolio of its pledgeable payoff, and consume the non-pledgeable payoff next period, which must be strictly positive in at least some state. This would imply infinite demand at some node and violate the market clearing conditions.

While (13) also holds in frictionless markets, (14) is specific to our model as it involves the parameter θ reflecting that trees' payoffs are imperfectly pledgeable. Taken together, the second and third no-arbitrage relationships show that, in our model, the law of one price may only fail in one direction: trees can be priced below, but not above, the portfolio of Arrow securities replicating their payoff. Below we show that strict violations of the law of one price arise in equilibrium.

[11] Notice that this reasoning only applies to trees in positive supply, which is why it only holds almost everywhere according to $\bar{N}$. For trees in zero supply, the only restriction is that the price must be large enough so that agents do not find optimal to hold them.

B. *Equilibrium Existence*

Establishing existence is challenging in part because some equilibrium objects are infinite-dimensional: tree portfolios are represented by finite measures and, correspondingly, tree prices are represented by continuous functions. Moreover, since prices enter incentive constraints, we cannot apply existence arguments based on welfare theorems (Negishi 1960). Instead, we use the classical price-player proof of Arrow and Debreu (1954), with two changes. First, since agents face incentive constraints that depend on prices, we must revisit the proof that constraint sets are lower hemicontinuous with respect to prices. Second, the constraint set of the price player must allow deviations from the law of one price and, correspondingly, its objective must account for the arbitrage revenues generated by agents' net trades in the market for trees (see Appendices A.1 and A.4). One advantage of the "cash on hand" formulation of budget and incentive constraints is to help coping with these difficulties.[12]

The proof of existence proceeds in two steps. We first consider tree supplies with finite support, a simpler case because it can be handled with finite-dimensional vector space methods. In particular, we can first determine finitely many tree prices, in the support of the supply distribution, and then provide a natural extension of this price vector to a continuous price function valuing all dividend streams in Δ. Next, we rely on the fact that the set of positive measures on Δ with finite support is dense in the set of all positive measures on Δ, endowed with the weak topology. Given a sequence of discrete measures converging weakly to any arbitrary finite measure $\bar{N}$, and an associated sequence of equilibria, we can extract a subsequence converging to an equilibrium given supply $\bar{N}$. In sum, we have the following result.

THEOREM 1: *There exists an equilibrium.*

While our analysis so far relied on the assumption that agents cannot short trees, it turns out that this constraint is not binding. Suppose indeed that, in addition to Arrow securities, agents are allowed to short trees: then agent i's tree portfolio can be written as the difference between two positive measures, $N_i = N_i^+ - N_i^-$, where N_i^+ represents long and N_i^- short positions. Since short positions are liabilities, they must be subject to some incentive constraint. Going through the same reasoning as before, we obtain

$$(15) \qquad a_{it+1}^-(s^t,s) + \int \left[\delta_{t+1}(s^t,s) + P_{t+1}(\delta|s^t,s)\right] dN_{it}^-(\delta|s^t)$$
$$\leq (1-\theta)\left\{a_{it+1}^+(s^t,s) + \int \left[\delta_{t+1}(s^t,s) + P_{t+1}(\delta|s^t,s)\right] dN_{it}^+(\delta|s^t,s)\right\}.$$

[12] Indeed, by suppressing the need to clear the market for Arrow securities this formulation makes it easier to formulate Walras law and define the price-player objective. Moreover, cash-on-hand can be used as state variable for a recursive proof of lower hemicontinuity.

We then establish the following result.

COROLLARY 1: *An equilibrium arising when agents can only short Arrow securities remains an equilibrium when agents can short both trees and Arrow securities.*

To see why the result obtains, consider an equilibrium when agents can only short Arrow securities. In equilibrium, as stated in Lemma 1, trees are priced below (but not above) replicating portfolios of Arrow securities. Hence, agents do not find it optimal to short trees: they prefer instead to short replicating portfolios of Arrow securities.[13]

Of course, while tree short selling constraints do not bind, incentive constraints could bind. We now examine conditions under which it is the case. Let (Q, c) be the price system and consumption allocation of a frictionless market equilibrium, i.e., with complete market and no incentive constraints. Now consider a corresponding economy with incentive constraints.[14] We say that (Q, c) is *IC-implementable* if there exists an equilibrium with incentive constraints, $(\hat{P}, \hat{Q})$ and $(\hat{c}, \hat{N})$ such that $Q = \hat{Q}$ and $c = \hat{c}$. In Appendix D we derive necessary and sufficient conditions for IC-implementability, leading to the next result.

PROPOSITION 1: *Let $\theta^\star$ be the largest θ such that a given frictionless market equilibrium is IC-implementable:*

(*i*) $\theta^\star > 0$ *if e is small and Inada conditions hold for all agents;*

(*ii*) $\theta^\star < 1$ *if $\bar{N}$ is small, $e \gg 0$ and marginal rates of substitutions evaluated at e are not equalized;*

(*iii*) $\theta^\star < 1$ *if agents have heterogeneous constant relative risk-aversion (CRRA) utility, e is small, and there is one tree;*

(*iv*) $\theta^\star$ *increases weakly if trees are unbundled into replicating portfolios.*

Notice that if an allocation is IC-implementable for a given θ, it remains IC-implementable if θ is lower.

The first two points of the proposition highlight that frictionless market outcomes obtain when the supply of collateral is sufficiently large and pledgeable.

Under the assumptions stated in the first point, agents want their cash-on-hand to remain large enough at every node: otherwise, since they do not have much labor income, they would be forced to consume little, which is not optimal under Inada conditions. This means that agents do not find it optimal to issue large liabilities.

[13] An earlier draft of this paper showed a stronger result. Namely, in a two-periods version of the model, any equilibrium with short selling is essentially equivalent to an equilibrium with no short-selling, with identical consumption allocation and price system.

[14] Formally, in a corresponding economy with incentive constraints, agents have the same preference and consumption-good endowment as in the complete-market economy, that is, at all nodes, the sum of their labor and tree income endowment is equal to their consumption endowment in the complete-market economy. Notice that there are many possible such economies, differing in terms of their pledgeability parameter and of the break down of consumption good endowment between labor and tree income.

Hence, incentive constraints are slack as long as trees are sufficiently pledgeable, i.e., for all θ small enough.

In the second point, intertemporal marginal rates of substitution evaluated at e are not equalized, so agents would benefit from risk sharing to smooth consumption. But such risk sharing is ruled out by the scarcity of collateral, $\bar{N}(\Delta) \simeq 0$.

The last two points of the proposition emphasize that the implementability of frictionless market outcomes not only depend on size and pledgeability of the collateral supply, but also on its distribution.

To gain intuition about the third point recall that, in a complete market equilibrium with heterogeneous CRRA utility, agents' consumption shares are not constant: they depend on the current realization of the aggregate endowment. But if there is just one tree and little labor income, aggregate resources are all bundled in a single tree. As a result, when agents trade the tree, they can only attain approximately constant consumption shares. Hence agents need to issue liabilities to attain their frictionless-market state-dependent consumption shares. Correspondingly, a frictionless market equilibrium is not IC-implementable as long as θ is close enough to one, i.e., as long as agents cannot issue much liabilities.

For the fourth point, suppose as a special case that $e = 0$, and that the distribution of tree supply is maximally dispersed. Specifically, assume that all trees in positive supply are Arrow securities, in the sense that they only pay dividend at one node. Then, it is clear that the frictionless market outcome is IC-implementable: all agents can synthesize their frictionless market equilibrium consumption profile by purchasing these Arrow trees only, while respecting market clearing in the aggregate. Thus, if one unbundled the entire tree supply into Arrow securities, then a frictionless market outcome would obtain. Our proof provides a formal definition of unbundling that allows to generalize this observations: it shows that IC-implementation becomes easier if trees are unbundled into replicating portfolios.

C. *First-Order Conditions*

We now state the first-order necessary and sufficient conditions for the agent's problem (the formal derivation is in Appendix AC). Let $\lambda_{it}(s^t) \geq 0$ denote the multiplier for the sequential budget constraint (7) at node (t, s^t) and $\mu_{it+1}(s^t, s) \geq 0$ the multiplier for the incentive constraint (8). The first-order conditions with respect to $c_{it}(s^t)$ and $W_{it+1}(s^t, s)$ write

$$\beta^t \pi_t(s^t) u_i'(c_{it}(s^t)) = \lambda_{it}(s^t), \tag{16}$$

$$\lambda_{it+1}(s^t, s) + \mu_{it+1}(s^t, s) = \lambda_{it}(s^t) Q_{t+1}(s^t, s), \tag{17}$$

where we have assumed strictly positive consumption for simplicity. The first condition states that the agent chooses consumption to equate marginal utility with marginal cost, which is equal to the multiplier of the budget constraint, $\lambda_{it}(s^t)$. The second condition equates the marginal value and marginal cost of increasing cash-on-hand next period, $W_{it+1}(s^t, s)$. It reveals that the marginal value of increasing cash-on-hand next period has two components: it relaxes both the budget constraint, with marginal value $\lambda_{it+1}(s^t, s)$, and the incentive constraint, with marginal value $\mu_{it+1}(s^t,s)$. The intuition for the latter is that higher cash-on-hand reduces the agent's incentive to default. Combining the two we obtain that

$$Q_{t+1}(s^t,s) \;=\; \beta\pi_{t+1}(s|s^t)\frac{u_i'\big(c_{it+1}(s^t,s)\big)}{u_i'\big(c_{it}(s^t)\big)} + \frac{\mu_{it+1}(s^t,s)}{\lambda_{it}(s^t)}. \tag{18}$$

Condition (18) is familiar from the limited-commitment literature (see, e.g., Alvarez and Jermann 2000), and means that Arrow securities are priced by those agents whose incentive constraints are not binding, for example, agents who are long Arrow securities.[15] When an agent's incentive constraint is not binding $\mu_{it+1}(s^t,s) \;=\; 0$ and the agent's intertemporal marginal rate of substitution is equal to the corresponding Arrow security price. By contrast, for the agents whose incentive constraint is binding, $\mu_{it+1}(s^t,s) \;>\; 0$ and the agent's intertemporal marginal rate of substitution is strictly lower than the corresponding Arrow security price. This, however, does not prompt the agent to sell the Arrow security because this would violate her incentive constraint.

New to our setting is the first-order condition with respect to tree holdings, which can be stated as

$$\begin{aligned}\lambda_{it}(s^t)\Big(-P_t(\delta|s^t) + \sum_s Q_{t+1}(s^t,s)\big[\delta_{t+1}(s^t,s) + P_{t+1}(\delta|s^t,s)\big]\Big)&\\ - \sum_s \mu_{it+1}(s^t,s)\theta\big[\delta_{t+1}(s^t,s) + P_{t+1}(\delta|s^t,s)\big] \;&\leq\; 0,\end{aligned} \tag{19}$$

with an equality if the agent holds the tree, that is, if $dN_{it}(\delta|s^t) \;>\; 0$. Substituting (18), we obtain

$$\sum_s Q_{it+1}(s^t,s)\big[\delta_{t+1}(s^t,s) + P_{t+1}(\delta|s^t,s)\big] \;\leq\; P_t(\delta|s^t), \tag{20}$$

where

$$Q_{it+1}(s^t,s) \;\equiv\; (1-\theta)\,Q_{t+1}(s^t,s) + \theta\beta\pi_{t+1}(s|s^t)\frac{u_i'\big(c_{it+1}(s^t,s)\big)}{u_i'\big(c_{it}(s^t)\big)} \tag{21}$$

is the *private* valuation of agent i for an Arrow security paying off in state s at time $t+1$. The economic interpretation is that the payoff has both a pledgeable component and a non-pledgeable component, which are valued differently. While the agent values the pledgeable component using the price of Arrow securities (first term on the right-hand side of (21)), it values the non-pledgeable component using its own intertemporal marginal rate of substitution (second term). To the extent that different agents' incentive constraints bind in different states, marginal rates of substitution and therefore private valuations differ across agents.

[15] By market clearing, there always exists an agent whose position is long.

D. *Segmentation*

Optimal Payoff Sets.—For each node (t, s^t), each agent i, and any vector of one-period ahead payoff $x \in \mathbb{R}_+^S$, the left-hand side of equation (20) defines a private valuation operator:

$$Q_{it+1}(s^t) \cdot x = \sum_s Q_{it+1}(s^t, s) x(s), \tag{22}$$

the dot product between the vector of private valuations for Arrow securities, and the vector of one-period-ahead payoffs. Correspondingly, there is a set of one-period ahead payoffs for which agent i is the best holder:

$$X_{it}(s^t) \equiv \left\{ x \in \mathbb{R}_+^S : Q_{it+1}(s^t) \cdot x \geq Q_{jt+1}(s^t) \cdot x, \text{ for all } j \right\}. \tag{23}$$

Agent i only holds trees whose vector of one-period ahead payoffs (i.e., the vector of the cum-dividend price) lies in $X_{it}(s^t)$. In what follows, we will refer to $X_{it}(s^t)$ as the *optimal payoff set* of agent i. We show below that, in general, because agents have different private valuations, they have distinct optimal payoff sets, and so hold different trees in equilibrium. Hence, the tree market is endogenously segmented.[16]

Characterizing the collection of optimal payoff sets, $\{X_{it}(s^t), i = 1, \ldots, I\}$, is a classical problem in optimal transport theory, studied in chapter 5 of Galichon (2016): the problem of drawing "power-diagrams." Although this problem does not have an analytical solution in general, it has simple geometrical properties. Moreover, numerical calculations are facilitated by the observation that optimal payoff sets solve a convex optimization problem. See online Appendix IX for more mathematical details. The proposition below states some key properties of optimal payoff sets.[17]

PROPOSITION 2: *The collection of optimal payoff sets* $\{X_{it}(s^t), i = 1, \ldots, I\}$, *has the following properties:*

(*i*) *Optimal payoff sets are convex polyhedral cones covering* $\mathbb{R}_+^S$.

(*ii*) *For any two pairs of agents, the optimal payoff sets are either identical* ($X_{it}(s^t) = X_{jt}(s^t)$) *or have disjoint interiors* ($\mathring{X}_{it}(s^t) \cap \mathring{X}_{jt}(s^t) = \emptyset$).

(*iii*) *If no incentive constraint binds in the next period, then* $X_{it}(s^t) = \mathbb{R}_+^S$ *for all i. Otherwise, if there exists an agent i whose incentive constraint binds in some state in the next period, then* $X_{it}(s^t)$ *is a strict subset of* $\mathbb{R}_+^S$ *and there exists another agent j such that* $\mathring{X}_{it}(s^t) \cap \mathring{X}_{jt}(s^t) = \emptyset$.

[16] Of course, for trees, one-period ahead payoffs depend on future prices and so are endogenous. In Section IIF, we show how to characterize segmentation in the set of dividend streams as opposed to the set of payoffs.

[17] The reader may wonder whether optimal payoff sets have other geometrical properties than the ones noted in Proposition 2 below. It turns out that these properties are only necessary: the geometry of power diagrams places additional restrictions on optimal payoff sets (see Aurenhammer 1987a, b; Ziegler 1995; and our summary in online Appendix IX) but these do not have obvious economic interpretations.

The first bullet point of the proposition follows because optimal payoff sets are defined by linear inequalities. The payoff vector of a tree is represented by a point in $\mathbb{R}_+^S$. The direction of the vector represents the tree's risk exposure, i.e., the distribution of its payoff across states. That optimal payoff sets are cones means that if an asset is in the set, any asset with proportional payoff, i.e., with the same risk exposure, is also in the set. To illustrate this, Figure 2 displays a convex polyhedral cone in the payoff space when there are three states. The rightmost facet of the polyhedron is the intersection of the cone with the simplex, which will be useful for the analysis.

The interpretation of the second bullet point is the following. If two agents have the same private valuations, $Q_{it+1}(s^t, s) = Q_{jt+1}(s^t, s)$ for all $s \in S$, then they must have the same optimal payoff sets. Otherwise, if two agents have different private valuations, the set of payoffs for which they have the same private valuations is an hyperplane; thus its interior is empty.

Finally, we turn to the third bullet point. If no incentive constraint binds, all private valuations are the same, so equation (23) implies that agents' have the same optimal payoff sets, which must be $\mathbb{R}_+^S$. If an incentive constraint binds for some agent i in some state s, it means agent i has large liabilities in state s, created by short positions in Arrow securities paying off in this state. By market clearing, there is another agent $j \neq i$ who has long positions in these Arrow securities and therefore no liabilities in state s. Thus, agent j's incentive constraint is slack in state s. Hence, agents i and j have different private valuations. In particular, agent i has a lower valuation than j for trees with large payoff in state s. Therefore, agents i and j have different optimal payoff sets and correspondingly different tree holdings; i.e., there is segmentation.[18]

To illustrate how segmentation in the market for trees is related to the demand for Arrow securities, consider agents who want to hedge against the risk of a given state s occurring. These agents purchase Arrow securities paying off in that state. But the supply of Arrow securities is limited by incentive constraints, hence insurance is imperfect and these agents have high marginal utility in that state and therefore high private valuations for trees with a high payoff in that state. Therefore, they buy trees lying in a cone that is close to the axis corresponding to that state.

While we have a characterization of optimal payoff sets given the vector of private valuations, it is difficult to solve analytically the general equilibrium problem of finding the private valuations. To sidestep this difficulty, in the parametric examples below, we consider the limiting case of an economy with no collateral, $\bar{N} = 0$.[19] In that case, the marginal rates of substitution are easy to characterize because the economy becomes autarkic: agents just consume their labor endowments. Equipped with those marginal rates of substitutions, we can solve the optimal transport

[18] Our result that private valuation operators differ across agents (so that tree holdings also differ) is driven by imperfect pledgeability of trees. If only labor income was imperfect pledgeable, agents would have the same private valuation operators, as can be seen in equation (21) for $\theta = 0$.

[19] The economy with no collateral does not satisfy our maintained assumption (1) that the aggregate dividend is strictly positive at all nodes, so Theorem 1 does not apply. However, it is easy to show by hand that an equilibrium exists, that the allocation is unique, and that an equilibrium price system is obtained from the same first-order conditions as in the rest of the paper. See online Appendix VIII.

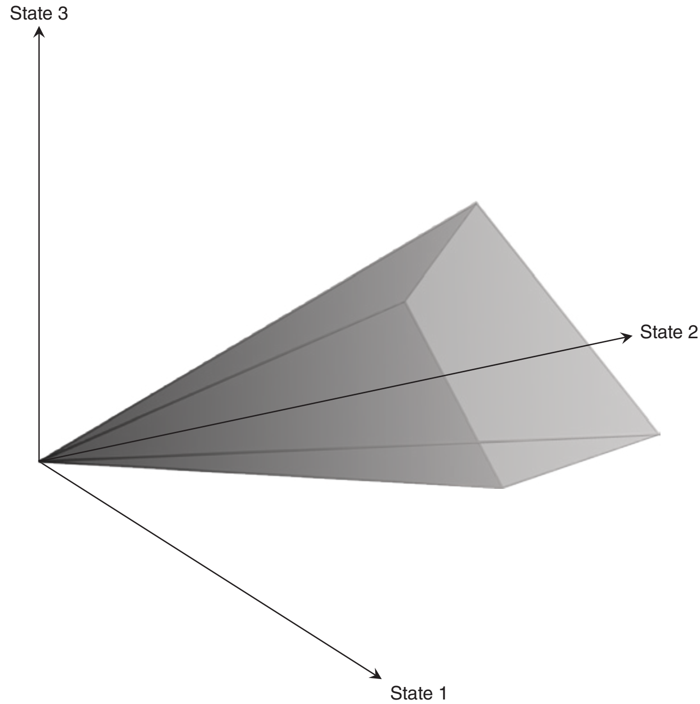

FIGURE 2. A POLYHEDRAL CONE WITH THREE SUCCESSOR STATES

problem and characterize the collection of optimal payoff sets.[20] Of course, when $\bar{N} = 0$, there are no trees around that agents can use as collateral to issue Arrow securities. But we establish a continuity argument in online Appendix VIII: the $\bar{N} = 0$ marginal rates of substitution, as well as the corresponding optimal payoff sets, approximate those arising in an economy with little collateral, $\bar{N} \simeq 0$.[21] This implies that, when $\bar{N} \simeq 0$, agents will purchase the trees whose payoffs lie in the interior of their $\bar{N} = 0$ optimal payoff set, and use them as collateral to sell Arrow securities whose $\bar{N} = 0$ price is strictly larger than their marginal rates of substitution.

A First Parametric Example.—We consider an economy populated by many log utility agents, who are hit by heterogeneous endowment growth shocks i.i.d. over time. Suppose there are three states (see online Appendix IX for assumptions and computations). To represent graphically the collection of optimal payoff sets, we plot their intersection with the simplex, as shown in Figure 3. These intersections fully characterize the optimal payoff sets, since these are cones. The figure reveals that some agents only hold assets near corners, i.e., assets that are approximately

[20] In an economy with no collateral $\bar{N} = 0$, borrowing constraints are "maximally tight" in the sense of Krussell, Mukoyama, and Smith (2011). As a result, the allocation becomes trivial: it is autarkic and agents consume "hand-to-mouth." This property makes an economy with no collateral as tractable as a representative agent economy: optimal payoff sets and asset prices can be characterized in closed form.

[21] To be clear, our arguments establish continuity for allocations and price systems, and upper hemicontinuity for optimal payoff sets.

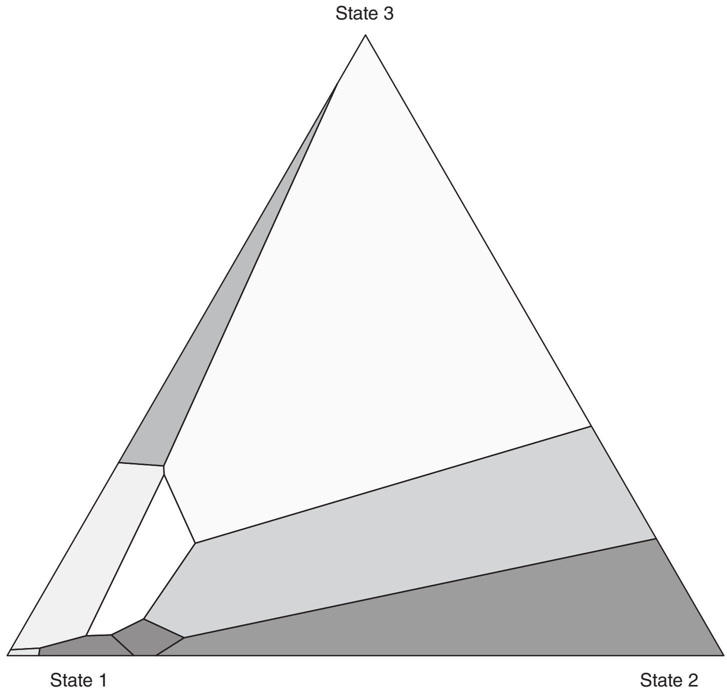

FIGURE 3. FIRST PARAMETRIC EXAMPLE: INTERSECTIONS OF OPTIMAL PAYOFF SETS WITH THE SIMPLEX

Arrow securities. These agents have the highest intertemporal marginal rates of substitutions corresponding to that state. Other agents hold assets away from corners. These agents do not have a maximal intertemporal marginal rate of substitution for any state, and so they do not hold any Arrow security. However, they can be the best holders of some other interior trees, with positive payoff in all states. In equilibrium, these agents buy these interior trees and use them as collateral to sell Arrow securities conditional on all states.

A Second Parametric Example.—Our second example provides a full characterization of equilibrium allocations in an economy log utility agents and with two states. We parameterize the endowment growth of agents by $\alpha_1 = 0 < \alpha_2 < \cdots < \alpha_I$. In state 1 occurring with probability π_1, the endowment growth is

$$g_1(\alpha) = g\left(1 + k_1\alpha\right)^{-\phi}, \tag{24}$$

while in state 2 occurring with probability π_2, it is

$$g_2(\alpha) = g\left(1 - k_2\alpha\right)^{-\phi}, \tag{25}$$

where $g > 0$, $0 < k_1\pi_1 < k_2\pi_2$, and $k_2\alpha_I < 1$. For all agents, $g_1(\alpha) \leq g_2(\alpha)$, meaning that $s = 1$ is the "bad state" while $s = 2$ is the "good state." Agents with higher α have a higher exposure to the risk of the bad state. The payoff of a tree in the simplex is $\left(x, 1 - x\right)$ where x is the payoff in the bad state.

PROPOSITION 3: *In the parametric example above, if* $\phi\gamma < 1$, *there exists a strictly increasing sequence* $x_0 = 0 < x_1 < \cdots < x_I = 1$ *such that the intersection of the optimal payoff set of agent* α_i *with the simplex is* $X_i = [x_{i-1}, x_i]$.

Therefore, agents with higher exposure to risk of the bad state (with higher α) hold trees with large payoffs in the bad state, which hedge them better. Extreme agents $\alpha_1 = 0$ and α_I hold assets in the neighborhood of Arrow securities, while intermediate agents hold other assets. Figure 4 represents the optimal payoff sets as shaded areas, in an example economy with $I = 5$ agents. The cones further to the northwest correspond to lower values of α. The sequence x_i is represented as the tick labels of the x-axis.[22]

E. *Asset Pricing*

From equation (20), it follows that the recursion

$$P_t(\delta | s^t) = \max_i \sum_s Q_{it+1}(s^t, s)\left[\delta_{t+1}(s^t, s) + P_{t+1}(\delta | s^t, s)\right] \tag{26}$$

defines an equilibrium price function for all trees $\delta \in \Delta$.[23] Equation (26) means that the price of the tree is the maximum of the private valuations of all agents for that tree. In this section we study the implications of this asset pricing formula.

Deviations from the Law of One Price.—Because the pricing operator (26) is convex and linearly homogeneous in payoffs, a tree must be priced below *any* replicating portfolio of long positions, comprised of trees or Arrow securities. For example, considering three trees with time $t+1$ payoffs x, y, and $z = x + y$, respectively, we have

$$\max_i Q_{it+1} \cdot z \leq \max_i Q_{it+1} \cdot x + \max_i Q_{it+1} \cdot y;$$

that is, z is valued below its replicating portfolio $x + y$. The inequality is strict if there is no agent who has the highest valuation for both x and y. This is stated more generally in the next proposition.

PROPOSITION 4: *Consider tree* δ *in node* (t, s^t) *and a replicating portfolio* M; *that is*,

$$\delta_{t+1}(s^t, s) + P_{t+1}(\delta | s^t, s) = \int \left[\delta'_{t+1}(s^t, s) + P_{t+1}(\delta' | s^t, s)\right] dM(\delta')$$

[22] Online Appendix X covers the other case. We show that when $\phi\gamma \geq 1$, assets are only held by extreme agents, α_1 and α_I. In that case, the optimal payoff sets of intermediate agents are either empty or singleton (i.e., have an empty interior), two properties consistent with Proposition 2.

[23] In all equilibria, this equation holds with equality for trees in positive supply, and otherwise with inequality. We assume from now on that it holds with equality for all trees in Δ, which is natural and without much loss of generality: indeed, this equation determines the equilibrium price of tree δ as soon as its supply outstanding is arbitrarily small.

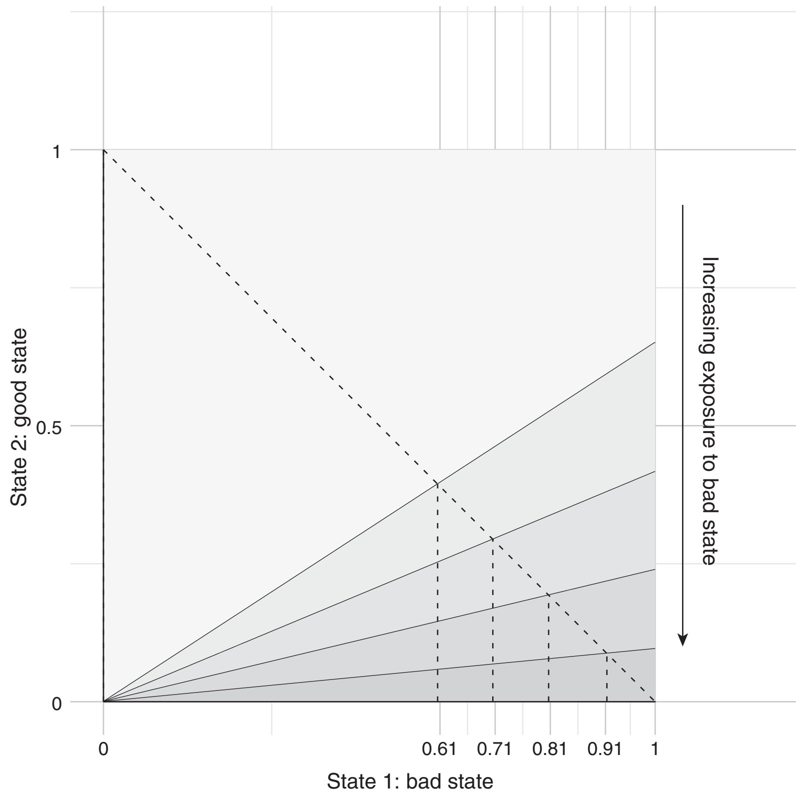

FIGURE 4. SECOND PARAMETRIC EXAMPLE: INTERSECTION OF OPTIMAL PAYOFF SETS WITH THE UNIT SQUARE

for all $s \in S$. If there exists no agent whose optimal payoff set includes the payoffs of (almost) all assets in the replicating portfolio, then tree δ is priced strictly below its replicating portfolio:

$$P_t(\delta|s^t) < \int P_t(\delta'|s^t)\, dM(\delta'). \tag{27}$$

Note that the replicating portfolio M can include trees, or Arrow securities, or both. The economic intuition of Proposition 4 is that a tree is a bundle of risks that cannot be traded separately from one another, whereas the portfolio of securities with the same payoff as the tree is a bundle of risks that can be traded separately. If there is no agent whose valuation for all the securities in that portfolio is the highest among agents, then no agent wants to hold all the securities in the replicating portfolio and bear the corresponding bundle of risks. Instead, all agents prefer to pick and choose among the risks in the bundle, retaining only those they want to bear. Therefore, the tree is priced below its replicating portfolio, that is, there is a basis.

A specific implication of our model is that the basis always goes in the same direction. Consistent with the no-arbitrage relationships in Lemma 1, the price of a tree can be lower than that of the replicating portfolio of trees and/or Arrow securities, but it cannot be higher. If it were higher, an agent holding the tree could sell it and buy the replicating portfolio. That arbitrage trade would be feasible because

(i) market clearing implies there is at least one agent holding the tree, and (ii) replacing a tree by its replicating portfolio does not tighten the incentive constraint. In contrast with (i), when the price of the tree is lower than that of the replicating portfolio, there does not exist an agent holding the replicating portfolio (since holding that portfolio is dominated). Hence arbitrage trades would require the issuance of liabilities, which would tighten the incentive constraint (in contrast with (ii)).

The basis is different from the collateral premium identified in the literature. Equation (21) shows that trees' payoffs can be decomposed into a pledgeable component and a non-pledgeable component, which are priced by different operators. Equation (18) implies that the pledgeable component is priced higher than the non-pledgeable one. In the Appendix, we derive the proofs for the general case in which trees have different pledgeabilities θ. In this context, trees whose cash flows are more pledgeable are priced higher. This is the collateral premium identified by the literature. In addition, we obtain the new result that there is a basis between a tree and a replicating portfolio of *identically* pledgeable securities. The basis reflects that the payoffs of the tree and the replicating portfolio are bundled differently across states.

For example, a convertible bond has the same cash flows as a straight bond and a call option on the issuer's stock. In the language of our model, a convertible bond is a tree with the same payoff as a combination of another tree (the straight bond) with a portfolio of Arrow securities (the call option). Our model implies that, if there are no agents who hold simultaneously the straight bond and the call, then the convertible bond should be priced strictly below the price of the straight bond plus the price of the call. In line with our theory, convertible bonds are in fact priced below the replicating portfolio. This deviation from the law of one price is at the root of a popular hedge fund strategy ("convertible arbitrage"), which consists in stripping the convertible bond (Mitchell and Pulvino 2012). Hedge funds buy the convertible bond and sell the set of securities that replicate the convertible bond to different clienteles: debt securities are distributed through prime brokers to money market funds and other buyers of safe securities, while equity risk is distributed to equity investors. The convertible arbitrage strategy is constrained, both in practice and in our theory, because the hedge funds realizing the arbitrage have a limited ability to sell the securities replicating the convertible bond. As a result, convertible bond cheapness increases when arbitrageurs have greater difficulties selling liabilities, such as during the 1998 Long-Term Capital Management crisis, the 2005 convertible arbitrage meltdown, and the 2008 credit crisis.

Concave Beta Pricing.—Denote tree return as $R_{t+1}(\delta|s^{t+1}) \equiv \left(\delta_{t+1}(s^{t+1}) + P_{t+1}(\delta|s^{t+1})\right)/P_t(\delta|s^t)$ and consider any K factors with expected returns normalized to zero. By linear projection onto the factors, we can write the tree returns as

$$R_{t+1}(\delta|s^{t+1}) \;=\; E_t\left[R_{t+1}(\delta|s^{t+1})\right] + \sum_{k=1}^{K}\beta_k(\delta|s^t)\,F_{k,t+1}(s^{t+1}) + \epsilon_{t+1}(\delta|s^{t+1}), \tag{28}$$

with

$$0 \;=\; E_t\left[F_{k,t+1}(s^{t+1})\right] \;=\; E_t\left[\epsilon_{t+1}(\delta|s^{t+1})\right] \;=\; E_t\left[F_{k,t+1}(s^{t+1})\,\epsilon_{t+1}(\delta|s^{t+1})\right]. \tag{29}$$

The next proposition states that if the factors orthogonalize tree returns with respect to all agents' private valuation operators (i.e., agents' stochastic discount factors), then a tree expected return is concave in the factor exposures.

PROPOSITION 5: *Let* $M_{it+1}(s^{t+1}) \equiv Q_{it+1}(s^{t+1})/\pi_{t+1}(s_{t+1}|s^t)$ *denote the stochastic discount factor for agent i. If*

$$E_t\left[M_{it+1}(s^{t+1})\,\epsilon_{t+1}(\delta|s^{t+1})\right] = 0 \quad \textit{for all } \delta \textit{ and } i, \tag{30}$$

then the expected return, $E_t\left[R_{t+1}(\delta|s^{t+1})\right]$, *is a concave function of factor exposures* $\left(\beta_k(\delta|s^t)\right)_{k=1}^K$.

Condition (30) holds in particular if the factors correspond to the stochastic discount factors of the different agents. The intuition for the concavity of expected returns in betas is similar to the intuition for the result that a tree is priced below a replicating portfolio of long positions in other trees (Proposition 4).[24] Consider a tree with a vector $\beta(\delta|s^t)$ of factor exposures. That tree can be replicated, up to unpriced risk $\epsilon_{t+1}(\delta|s^{t+1})$, with a portfolio of long positions in other trees such that a linear combination of their factor exposures are equal to $\beta(\delta|s^t)$. In line with Proposition 4, Proposition 5 states that the replicated tree has lower price and higher expected return than the replicating portfolio.

Our result above differs from the seminal analysis of Black (1972) on capital asset pricing with leverage constraints. Indeed, in the model of Black, the concavity results only holds for *efficient* portfolios, and the expected return of individual securities remains linear in beta. In our setting, by contrast, the expected returns on *individual* securities is concave in beta, in the spirit of empirical evidence of Frazzini and Pedersen (2014) and Hong and Sraer (2016).

F. *Optimal Payoff Sets versus Optimal Tree Sets*

In the analysis above, we have studied segmentation and asset pricing in terms of one-period ahead payoffs. However, these payoffs are in general endogenous since they depend on next-period prices. We now show that our analysis applies to the set of exogenous dividend streams instead of endogenous one-period-ahead payoffs, and discuss new insights.

When viewed as a Bellman equation, the recursive pricing formula (26) reveals that tree prices solve an intertemporal optimization problem: find the state-contingent sequence of asset holders with the highest valuation for the tree. More formally, define a sequence of agents as $J = \left\{j_t(s^t) \in \{1,\ldots,I\} : 0 \le t < T, s^t \in S^t\right\}$, and let $\mathcal{J}$ denotes the set of all such sequences. Any sequence J generates a private-valuation for node (t, s^t) consumption defined recursively as

$$q_{t+1}(s^{t+1}|J) = q_t(s^t|J) \times Q_{j_t(s^t),t+1}(s_{t+1}|s^t),$$

[24] Both results are driven by imperfect pledgeability of assets and would not hold for $\theta = 0$.

with the convention that $q_0(s^0|J) = 1$ for all J. Then, a standard optimality verification argument (in online Appendix X.1) shows that

$$P_t(\delta|s^t) = \max_{J \in \mathcal{J}} \sum_{(u,s^u) \succ (t,s^t)} \frac{q_u(s^u|J)}{q_t(s^t|J)} \delta_u(s^u). \tag{31}$$

This formula reveals that the pricing function is the maximum of a family of linear functions of δ, each corresponding to a particular sequence of agents, J. Hence our earlier characterization of optimal payoff sets carries over to trees, but with one key difference: instead of characterizing the optimal *payoff* set of an *individual* agent, j, the optimal transport problem now characterizes the optimal *tree* set of a *sequence* of agents. The optimal tree set of agent j is the union of optimal tree sets over all sequences J of agents starting with agent j. Hence, the optimal tree set of agent j is not necessarily convex.

Now turning to pricing, one immediate implication of (31) is that the pricing function is piecewise linear and convex in dividend streams δ. Another implication, that has been highlighted in prior work on asset pricing with heterogeneous beliefs (Harrison and Kreps 1978, Scheinkman and Xiong 2003), is that the function prices the option to resell the tree to a different type of agent at a later date. Formally, the price of the tree has to be greater than the buy-and-hold value that would be derived by a constant sequence of agents, and strictly so if the optimum of (31) is not attained by a constant sequence.

III. Conclusion

This paper offers a dynamic general equilibrium analysis of risk sharing and asset pricing when collateral is imperfectly pledgeable. This yields a rich set of implications on asset holdings (endogenous segmentation) and asset pricing (basis, concavity of expected returns in factor loadings).

While our model is set in the context of a pure exchange economy, it would be interesting to extend the analysis to study investment in production technologies. In our theoretical framework, these technologies have a dual role: they expand production possibilities but, at the same time, they can be used as collateral and thus improve risk-sharing and consumption smoothing. One would expect that, in some cases, there would be a trade-off between technological and collateral efficiency, breaking standard separation results in classical theories. That is, a technology that leads to a large increase in production efficiency may be a poor collateral, and vice versa. The trade-off between collateral and technological efficiency may have implications for international trade and for the nexus between technological and financial development.

Appendix

We establish several of our results in the extended environment with a pledgeability parameter that is agent and tree specific. That is, we assume that each agent i can pledge a fraction $\theta_i(\delta)$ of the next-period payoff of tree $\delta \in \Delta$, where $\theta_i(\delta) \in (0,1)$ for all i and δ, is continuous and uniformly bounded away from zero and one across all agents and trees.

In what follows we will use the following notations and definitions. For each (t, s^t), we let $\Delta_t^+(s^t)$ denote the set of trees with nonzero continuation dividend, i.e., such that $\delta_u(s^u) > 0$ for some $(u, s^u) \succ (t, s^t)$. The set of functions continuous on Δ is endowed with the metric induced by the sup norm. The set of positive and finite measures over Δ is denoted by $\mathcal{M}_+$ and is endowed with the topology of weak convergence.[25]

Finally, as is standard, it is easier to establish equilibrium existence when prices are deflated to time zero. Namely, after fixing the price of consumption a time zero to be $q_0(s^0)$, we let the deflated or time-zero price of consumption at node (t, s^t) to be $q_t(s^t) \equiv q_0(s^0) Q_1(s^0, s_1) Q_2(s^1, s_2) \cdots Q_t(s^{t-1}, s_t)$. Likewise, the deflated price of trees is given by the function $p_t(\delta|s^t) \equiv q_t(s^t) P_t(\delta|s^t)$. For the remainder of this Appendix, we work with this deflated price system (p, q). Correspondingly, we write the sequential budget constraint as

$$\begin{aligned} q_t(s^t)\, c_{it}(s^t) &+ \int p_t(\delta|s^t)\, dN_{it}(\delta|s^t) + \sum_s q_{t+1}(s^t, s) W_{it+1}(s^t, s) \\ &= q_t(s^t)\, W_{it}(s^t) + \sum_s q_{t+1}(s^t, s) e_{it+1}(s^t, s) \\ &\quad + \sum_s \int \left[q_{t+1}(s^t, s)\, \delta_t(s^t, s) + p_{t+1}(\delta|s^t, s)\right] dN_{it}(\delta|s^t, s), \end{aligned} \tag{32}$$

for all (t, s^t), and with the convention that time $T + 1$ variables and time T tree prices are equal to zero. Likewise, we write the incentive constraint as

$$\begin{aligned} &q_{t+1}(s^t, s)\, W_{it+1}(s^t, s) \\ &\geq q_{t+1}(s^t)\, e_{it+1}(s^t, s) + \theta \int \left[q_{t+1}(s^t, s)\, \delta_{t+1}(s^t, s) + p_{t+1}(\delta|s^t, s)\right] dN_{it}(\delta|s^t). \end{aligned} \tag{33}$$

Appendix A. Preliminary Results

A. *Normalized No-Arbitrage Price Systems*

We let $\mathcal{NA}$ denote the set of normalized no-arbitrage price systems, that is, the set of (p, q) such that consumption prices q lie in the simplex:

$$\sum_{(t, s^t)} q_t(s^t) = 1, \tag{34}$$

[25] Recall that a sequence $N^\ell \in \mathcal{M}_+$ converges weakly towards some $N \in \mathcal{M}_+$ if $\int f(\delta)\, dN^\ell(\delta) \to \int f(\delta)\, dN(\delta)$ for all functions f continuous on Δ. We use sequences instead of nets to define weak convergence because $\mathcal{M}_+$ is metrizable (Varadarajan 1958). The same sequential characterization of weak convergence is used by Stokey and Lucas (1989).

tree prices $p_t(\delta|s^t)$ are continuous in $\delta \in \Delta$ for all (t,s^t), and the no-arbitrage conditions hold:

$$(35) \qquad q_t(s^t) > 0,$$

$$(36) \qquad p_t(\delta|s^t) \leq \sum_s \left[q_{t+1}(s^t,s)\,\delta_{t+1}(s^t,s) + p_{t+1}(\delta|s^t,s)\right],$$

$$(37) \qquad p_t(\delta|s^t) \geq \left(1-\theta_i(\delta)\right)\sum_s \left[q_{t+1}(s^t,s)\,\delta_{t+1}(s^t,s) + p_{t+1}(\delta|s^t,s)\right],$$

where (36) must hold for all δ, and (37) must hold for all agents i and trees δ, with a *strict inequality* if the continuation dividend is nonzero, $\delta \in \Delta_t^+(s^t)$.

Notice that we restrict attention here to price systems such that (36) holds for all $\delta \in \Delta$. Although this is stronger than the necessary condition for no-arbitrage of Lemma 1, this will not create problems for establishing existence. As will become clear, in any equilibrium, there always exists a price system such that (36) holds for all $\delta \in \Delta$. A useful property to keep in mind, shown in online Appendix I, follows.

LEMMA A.1: *The closure of $\mathcal{NA}$, denoted by $\overline{\mathcal{NA}}$, is the set of price systems (p,q) such that (35)–(37) hold with weak inequalities.*

B. *Properties of the Constraint Correspondence*

The constraint correspondence is the set-valued function $\Gamma_i(W_0,p,q)$ mapping any initial cash-on-hand $W_0 \in \mathbb{R}_+$ and price system $(p,q) \in \overline{\mathcal{NA}}$ to the set of plans (c_i,N_i) that are budget feasible and incentive compatible given the initial wealth W_0 and price system (p,q). An important property for what follows is defined below.

PROPOSITION A.1: *The constraint correspondence is nonempty, convex valued, has a closed graph over $\mathbb{R}_+ \times \overline{\mathcal{NA}}$, and is lower hemicontinuous over $\mathbb{R}_+ \times \mathcal{NA}$.*

The challenging part of the proof, shown in online Appendix II, is to establish lower hemicontinuity. To understand why, recall that $\Gamma_i(W_0,p,q)$ is lower hemicontinuous at (W_0,p,q) if for all $\left(W_0^\ell,p^\ell,q^\ell\right) \to (W_0,p,q)$, and for all $(c,N) \in \Gamma_i(W_0,p,q)$, there exists a sequence $\left(c^\ell,N^\ell\right) \to (c,N)$ such that, for all ℓ large enough, $\left(c^\ell,N^\ell\right) \in \Gamma_i\left(W_0^\ell,p^\ell,q^\ell\right)$ (see Section 3.3. in Stokey and Lucas 1989). Without incentive constraints, the proof of lower hemicontiniuity is easy. Indeed, if $c \neq 0$, then for $\left(p^\ell,q^\ell\right) \to (p,q)$, one can finance some $\left(c^\ell,N^\ell\right) \in \Gamma_i\left(W_0^\ell,p^\ell,q^\ell\right)$ which is arbitrarily close to (c,N) by reducing consumption slightly in some state. But this no longer works with our incentive constraints: indeed reducing consumption in some state corresponds to a reduction in the amounts of cash-on-hand planed for this state, which tightens incentive constraints.

Next, we define, for any $(p,q) \in \mathcal{NA}$, the demand correspondence of agent i: $Z_i(p,q) = \operatorname{argmax} U_i(c_i)$ with respect to $(c_i,N_i) \in \Gamma_i(W_0,p,q)$, where W_0 is defined by (9). An important step in the classical proof of existence is to show that the excess demand becomes arbitrarily large when the price system nearly violates the no-arbitrage conditions.

PROPOSITION A.2: *Take any* $(p,q) \in \overline{\mathcal{NA}} \backslash \mathcal{NA}$. *Then there exists some i such that, for any sequence* (p^ℓ, q^ℓ) *in* $\mathcal{NA}$ *converging to* (p,q), *any sequence of optimal demand* $(c_i^\ell, N_i^\ell) \in Z_i(p^\ell, q^\ell)$ *is unbounded in consumption.*

The intuition is that, by our maintained assumptions and the no-arbitrage conditions, agents' initial tree wealth must be bounded away from zero at all $(p,q) \in \overline{\mathcal{NA}} \backslash \mathcal{NA}$. This allows the agents who are best able to pledge (with low $\theta_i(\delta)$) to take advantage of near arbitrage opportunities as $(p^\ell, q^\ell) \to (p,q)$ and guarantee unbounded consumption. The detailed proof is in online Appendix III.

C. *First-Order Necessary and Sufficient Conditions*

We state first-order conditions when agents maximize taking as given the deflated price system (p,q), and subject to the corresponding budget and incentive constraints, (32) and (33). Without loss, we state the sequential budget constraints with weak inequalities. This allows us to apply Theorem 1 of chapters 8.3 and 8.4 in Luenberger (1969). In particular, since there is only a finite number of parametric constraints, the interior-point condition for the positive cone of the range of the parametric constraints is immediately satisfied. We thus obtain the following result.

PROPOSITION A.3: *Let* (p,q) *be a price system satisfying the no-arbitrage conditions of Lemma 1. Then the plan* (c_i, N_i) *solves agent i's problem if and only if it is budget and incentive feasible and there exist positive multipliers* $\hat{\lambda} = \{\hat{\lambda}_{it}(s^t), t \geq 0, s^t \in S^t\}$ *and* $\hat{\mu} = \{\hat{\mu}_{it}(s^t), t \geq 1, s^t \in S^t\}$ *such that*

$$\beta^t \pi_t(s^t)\, u_i'(c_{it}(s^t)) \leq \hat{\lambda}_{it}(s^t)\, q_t(s^t), \qquad \textit{with equality if } c_{it}(s^t) > 0, \tag{38}$$

$$\hat{\lambda}_{it}(s^t) = \hat{\lambda}_{it+1}(s^{t+1}) + \hat{\mu}_{it+1}(s^{t+1}), \tag{39}$$

$$\hat{\mu}_{it}(s^t)\Big\{ q_t(s^t)\, W_{it}(s^t) - q_t(s^t)\, e_{it}(s^t) - \int \theta_i(\delta)\big[q_t(s^t)\,\delta_t(s^t) + p_t(\delta|s^t)\big]\, dN_{it-1}(s^t)\Big\} = 0, \tag{40}$$

$$\int \big[\hat{v}_{it}(\delta|s^t) - p_t(\delta|s^t)\big]\, dM(\delta) \leq 0 \qquad \textit{for all } M \in \mathcal{M}_+ \textit{ with equality if } M = N_{it}(s^t), \tag{41}$$

where, for all $t < T$,

$$\hat{v}_{it}(\delta|s^t) \equiv \sum_s \left(1 - \frac{\theta_i(\delta)\, \hat{\mu}_{it+1}(s^t,s)}{\hat{\lambda}_{it}(s^t)}\right)\big[q_{t+1}(s^t,s)\,\delta_{t+1}(s^t,s) + p_{t+1}(\delta|s^t,s)\big].$$

Online Appendix IV provides the proof and derives the corresponding first-order conditions for the nondeflated price system (P,Q) shown in the text.

D. *Walras Law*

The classical proof of existence adds to the economy a fictitious player who sets price in order to maximize the value of excess demand—increasing the price of goods in positive excess demand and vice versa. In order to make an educated guess about the specification of the price player's objective in our setting, we now derive Walras law—that is, we calculate the value of the excess demand when all agents satisfy their budget constraints. The proof is in online Appendix V.

PROPOSITION A.4: *Suppose an allocation $(c,N)_{i\in I}$ is budget feasible for all i given some no-arbitrage price system (p,q). Then*

$$\sum_{(t,s^t)} q_t(s^t)\Big(\sum_i c_{it}(s^t) - \omega_t(s^t)\Big) - \sum_{(t,s^t),t<T} \int b_t(\delta|s^t)\Big(\sum_i dN_{it}(\delta|s^t) - d\bar{N}(\delta)\Big) = 0, \tag{42}$$

where $\omega_t(s^t) \equiv \sum_i e_{it}(s^t) + \int \delta_t(s^t) d\bar{N}(\delta)$ is the aggregate endowment and $b_t(\delta|s^t)$ is the basis function generated by the price system (p,q) at node (t,s^t), $t < T$:

$$b_t(\delta|s^t) = \sum_{s'}\big(q_{t+1}(s^t,s)\,\delta_{t+1}(s^t,s) + p_{t+1}(\delta|s^t,s)\big) - p_t(s^t), \tag{43}$$

with the convention that $p_T(s^T) = 0$.

Without incentive constraints, there are no bases and the second term is equal to zero, because all that matters in agents' intertemporal constraint is the present value of the dividend generated by the tree. In that case, Walras law takes its standard form. In our setting with incentive constraints, Walras law takes a non standard form for two reasons, giving rise to the second term in our expression of Walras law. First, agents generate extra revenue with basis trades: purchases of trees financed by sales of replicating portfolios of Arrow securities. Second, the bases depress the value of agents' initial tree endowment.

Appendix B. Proof of Theorem 1

A. *Existence of Equilibrium with Finitely Many Trees*

We first provide an existence proof when the tree supply has finite support, and when agents can only choose portfolios in that support; that is, we replace Δ by some finite subset, $\{\delta_1,\dots,\delta_K\} \subseteq \Delta$, and we choose a supply vector $\bar{N} \in \mathbb{R}^K_+$ such that the aggregate dividend is strictly positive in all states. Notice that, in this market setting, agents are restricted to choose portfolios in the finite subset $\{\delta_1,\dots,\delta_K\}$ instead of in Δ. Hence, portfolios are no longer finite measures but simply vectors, specifying the share holding of each tree k. Correspondingly we use the notation $n_{it}(s^t) \in \mathbb{R}^K_+$ to denote the portfolio chosen by agent i at node (t,s^t). Likewise, asset prices at node (t,s^t) are no longer represented by a function but by a vector $p_t(s^t) \in \mathbb{R}^K_+$. Finally, with some abuse of notation, $\delta_t(s^t) = (\delta_{1t}(s^t),\delta_{2t}(s^t),\dots,\delta_{Kt}(s^t)) \in \mathbb{R}^K_+$ now denotes the vector of all tree dividends at node (t,s^t).

The Fixed-Point Problem.—We now adapt the classical fixed-point problem to our setting. First, we let B denote the set of (c,n) such that

$$0 \leq c_t(s^t) \leq 2\sum_{(u,s^u)} \omega_u(s^u), \tag{44}$$

$$0 \leq n_t(s^t) \leq 2\bar{N}, \tag{45}$$

where $\omega_t(s^t)$ is the aggregate endowment, as defined in Proposition A.4. We let $Z_i^B(p,q)$ denote the demand correspondence of agent i when she is artificially constrained to choose $(c_i,n_i) \in \Gamma_i(W_0,p,q) \cap B$, with W_0 defined according to (9). This correspondence now has compact values. It continues to have a closed graph. We verify in online Appendix V.1 that it also continues to be lower hemicontinuous.

For any $0 < \varepsilon < \min_{i,k}\theta_{ik}$, we let $\mathcal{NA}^\varepsilon$ denote the set of $(p,q) \in \mathcal{NA}$ satisfying

$$q_t(s^t) \geq \varepsilon, \tag{46}$$

$$p_{kt}(s^t) \geq (1-\theta_{ik}+\varepsilon)\sum_s \big(q_{t+1}(s^t,s)\,\delta_{kt+1}(s^t,s') + p_{kt+1}(s^t,s')\big), \tag{47}$$

for all (t,s^t), all trees k, and all agents i. Note that $\mathcal{NA}^\varepsilon$ is nonempty since it contains price systems with zero basis. We use Walras law shown in Proposition A.4 to formulate the objective of the price player. Given any profile of consumption and tree portfolio choices $(c_i,n_i)_{i\in I}$, we let the price player's problem be $Z_0^\varepsilon(c,n) = \text{argmax} q\cdot\left(\sum_i c_i - \omega\right) - \mathbf{b}(p,q)\cdot\left(\sum_i n_i - \bar{N}\right)$, with respect to $(p,q) \in \mathcal{NA}^\varepsilon$ and where $\mathbf{b}(p,q)$ is the vector of bases generated by (p,q): $\mathbf{b}_t(s^t|p,q) \equiv \sum_s \big(q_{t+1}(s^t,s)\,\delta_{t+1}(s^t,s) + p_{t+1}(s^t,s)\big) - p_t(s^t)$, for each (t,s^t), $t<T$, with the convention that $p_T(s^T) = 0$. Next, we define the correspondence $\Psi : \mathcal{NA}^\varepsilon \times B^I \to \mathcal{NA}^\varepsilon \times B^I$ by

$$\Psi^\varepsilon\big(p,q,(c_i,n_i)_{i\in I}\big) = Z_0^\varepsilon(c,n) \times Z_1^B(p,q) \times \cdots \times Z_I^B(p,q).$$

We show in Appendix V.2 that the following result holds.

PROPOSITION B.1: *The correspondence Ψ^ε has a fixed point which, for ε small enough, is the basis of an equilibrium.*

The argument is very similar to the one of the classical proof. The key difference is that there are bases, which create a second term in the price player's objective, that the price player can only manipulate by choosing prices respecting no-arbitrage conditions. Hence, relative to the classical proof, the price player faces different objective and constraints, which makes it more difficult to establish that all markets clear.

B. *Existence of Equilibrium When the Supply Has a Finite Support*

The case of a finite support is almost identical to the case of finitely many trees. The only difference is that agents have a larger choice set: instead of being restricted to hold trees in $\{\delta_1, \delta_2, \dots, \delta_K\}$, they can choose to hold any tree in Δ: formally, instead of choosing vectors $n \in \mathbb{R}_+^K$ representing their holdings of each tree in the support $\{\delta_1, \delta_2, \dots, \delta_K\}$, they now choose positive finite measures $N \in \mathcal{M}_+$ representing their holdings of each tree in Δ. Correspondingly, the equilibrium must price all trees in Δ, and not only the trees in the support $\{\delta_1, \delta_2, \dots, \delta_K\}$. The following Lemma, proved in Appendix V.3, establishes that it is possible to choose a continuous price function, defined over Δ, such that the equilibrium identified in the previous section remains unchanged when agents can choose to hold any tree in Δ.

LEMMA B.1: *Consider any equilibrium price system* (p, q) *and allocation* (c, n) *with a finite number of trees* $\{\delta_1, \dots, \delta_K\}$ *in supply* $\bar{N} = (n_1, \dots, n_K)$. *Then, there exists an extension* $\hat{p}$ *of the price vector* p *to the set* Δ, *such that*

(i) *the function* $\hat{p}$ *belongs to an equicontinuous family* $\mathcal{P}$;

(ii) *the price system* $(\hat{p}, q)$ *belongs to* $\mathcal{NA}$;

(iii) *demands remain optimal when agents can choose to hold any tree in* Δ, *given* $(\hat{p}, q)$.

The price function $\hat{p}$ is obtained by taking first-order conditions, using the formula given Proposition A.3, and has a simple interpretation: it is the maximum marginal valuation of future payoffs, across all agents. It is intuitive that it belongs to an equicontinuous family, since it can be written as a function that is jointly continuous with respect to the dividend stream and the intertemporal marginal rates of substitutions of finitely many agents across finitely many periods. See Appendix V.3 for the precise argument.

C. *Existence of Equilibrium for an Arbitrary Supply*

Let $\bar{N}$ be an arbitrary positive finite measure over Δ. By the density Theorem 15.10 in Aliprantis and Border (2006), easily extended to the case of finite measures, there exists a sequence of measures $\bar{N}^\ell$ with finite support converging weakly towards $\bar{N}$. Invoking Lemma B.1, for each ℓ, there exists an equilibrium $(p^\ell, q^\ell, c^\ell, N^\ell)$. But this sequence of equilibria remains trapped in a compact set.[26] This allows to extract a subsequence converging weakly to some (p, q, c, N) (and keep the same notations).

By passing to the limit in the market-clearing conditions, one finds that the limiting consumption and portfolios are feasible. By passing to the limit in the budget and incentive constraints, one obtains that the limiting consumptions and portfolios

[26]Indeed the p^ℓ belongs to $\mathcal{P}$, an equicontinuous family of functions; the vector of consumption price q^ℓ is bounded by one; the vector of consumption c^ℓ is bounded by the aggregate endowment; and tree-market clearing ensures that each measures N_i^ℓ is bounded by $\bar{N}^\ell(\Delta) \leq 2\bar{N}(\Delta)$ for all ℓ large enough which, together with Theorem 15.11 in Aliprantis and Border (2006), ensures weak compactness.

belong to the constraint set of each agents: $(c_i, N_i) \in \Gamma_i(W_{i0}, p, q)$ for each i, where W_{i0} is defined according to (9).

What remains to be shown is that the limiting consumptions and portfolios are optimal. To that end, we first note that the limiting price system (p,q) must belong to $\mathcal{NA}$: otherwise, if instead $(p,q) \in \overline{\mathcal{NA}} \backslash \mathcal{NA}$, then by Proposition A.2, easily extended to account for the fact that agents' tree endowment $\alpha_i \bar{N}^\ell$ varies along the sequence, the consumption of some agent would grow unbounded as $\ell \to \infty$, which would contradict market clearing. By Proposition A.1, it follows that the constraint correspondence of each agent is lower hemicontinuous at (W_{i0}, p, q), where W_{i0} is defined according to (9). Now consider any agent i and any $(\hat{c}_i, \hat{N}_i) \in \Gamma_i(W_{i0}, p, q)$, where W_{i0} is defined according to (9). Lower hemicontinuity ensures that there exists some sequence $(\hat{c}_i^\ell, \hat{N}_i^\ell) \in \Gamma_i(W_{i0}^\ell, p^\ell, q^\ell)$, where W_{i0}^ℓ is defined according to (9), such that $(\hat{c}_i^\ell, \hat{N}_i^\ell)$ converges towards $(\hat{c}, \hat{N})$. Clearly, for each ℓ, $U_i(c_i^\ell) \geq U_i(\hat{c}_i^\ell)$, where c_i^ℓ is the optimal consumption demand in the equilibrium given the equilibrium (p^ℓ, q^ℓ) identified above. Passing to the limit, we obtain $U_i(c_i) \geq U_i(\hat{c}_i)$, which means that (c_i, N_i) is indeed optimal given (p,q).

Appendix C. Proof of Corollary 1

See online Appendix VI.

Appendix D. Proof of Proposition 1

We start by deriving a necessary and sufficient condition for a complete market equilibrium to be IC-implementable. The proof is in online Appendix VII.

LEMMA D.1: *A frictionless market equilibrium (q, c) is IC-implementable if and only if there exists a feasible tree allocation N such that*

$$\sum_{(u,s^u) \succeq (t,s^t)} q_u(s^u)\big(c_{iu}(s^u) - e_{iu}(s^u)\big) \geq \theta \int \left[q_t(s^t)\,\delta_t(s^t) + p_t(\delta|s^t)\right] dN_{it-1}(\delta|s^{t-1}), \tag{48}$$

for all agents and at all nodes following time zero, that is for all i, all (t, s^t), $t \geq 1$.

Point 1: If Inada conditions are satisfied then, in a frictionless market equilibrium the consumptions of all agents are bounded away from zero at all nodes. If in addition labor income is small enough, then the left-hand side of (48) is strictly positive. It then follows that (48) holds with the feasible tree allocation $N_i = \bar{N}/I$ and θ small enough.

Point 2: Consider a frictionless market equilibrium (c, q) when agents have heterogeneous CRRA utility. Because of complete market and no frictions, it follows that consumption of agent i can be written as a function of the aggregate endowment only, $c_{it}(s^t) = y_t(s^t) f_i[y_t(s^t)]$, where f_i denotes the consumption share. Because of heterogeneity in CRRA utility it follows that the consumption share function, f_i, is strictly increasing in y for the least risk averse agent.

Now assume, toward a contradiction, that this equilibrium is implementable with just one tree for arbitrarily small labor income, and pledgeability parameter arbitrarily close to one. That is, there exists a sequence of labor endowment, $e^p \to 0$, of pledgeability parameter $\theta^p \to 1$ such that the frictionless market equilibrium is IC-implementable for all p. Using (48) in the last period, this means that there exist some tree holdings $n_{iT-1}(s^{T-1})$ such that

$$c_{iT}(s^T) - e^p_{iT}(s^T) \geq \theta^p n^p_{iT-1}(s^{T-1})\, \delta^p_T(s^T),$$

where $\delta^p_T(s^T)$ is the dividend of the tree and $\sum_i n^p_{iT}(s^T) = 1$, (normalizing the aggregate supply of the tree to one). Dividing though by the aggregate endowment, this gives

$$f_i\big[y_T(s^T)\big] \geq \theta^p n^p_{iT-1}(s^{T-1})\frac{\delta^p_T(s^T)}{y_T(s^T)} + \frac{e^p_{iT}(s^T)}{y_T(s^T)}.$$

Given that $n^p_{iT-1}(s^{T-1}) \in [0,1]$, we can extract subsequences for each i converging to $n_i^\star$ such that $\sum n_i^\star = 1$. Moreover, since the labor endowment $e^p \to 0$, it follows that the tree dividend $\delta^p_T(s^T) \to y_T(s^T)$. With this in mind, when we go to the limit, we find $f_i\big[y_T(s^T)\big] \geq n_i^\star$, for all i. But since $\sum_i f_i\big[y_T(s^T)\big] = \sum_i n_i^\star = 1$, it follows that $f_i\big[y_T(s^T)\big] = n_i^\star$ for all i, which is a contradiction since we noted earlier that the consumption share function $f_i(y)$ is strictly increasing for the least risk-averse agent.

Point 3: Fix some strictly positive labor income $e \gg 0$ such that agents' intertemporal marginal rates of substitutions are not equalized when evaluated at e_i. Toward a contradiction, consider a sequence $\bar{N}^\ell$ such that $\bar{N}^\ell(\Delta) \to 0$, and a corresponding sequence of equilibria $(p^\ell, q^\ell, c^\ell, N^\ell)$ such that (q^ℓ, c^ℓ) is a frictionless market equilibrium for all ℓ. Proposition VIII.2 in the online Appendix shows that $(p^\ell, q^\ell, c^\ell, N^\ell)$ converges to the unique equilibrium that obtain when $\bar{N}(\Delta) = 0$, what Proposition VIII.1 in the online Appendix calls a "zero-collateral equilibrium." In particular $c^\ell_{it}(s^t)$ converges to $e_{it}(s^t)$. It follows that consumptions are strictly positive and intertemporal marginal rates of substitution are not equalized, implying that the equilibrium allocation cannot coincide with that of a frictionless market equilibrium, a contradiction.

Point 4: Consider two distributions of tree supplies $\bar{N}'$ and $\bar{N}$. Then, we say that *$\bar{N}'$ unbundles the payoffs of the trees in $\bar{N}$ into replicating portfolios* if there exists a transition probability function $d\Omega(\delta'|\delta)$ (see Definition 8.1 in Stokey and Lucas 1989) representing the weight of tree δ' in a replicating portfolio for tree δ, such that two conditions are satisfied. First, the portfolio $d\Omega(\delta'|\delta)$ must replicate δ:

$$\int_{\delta'} \delta' d\Omega(\delta'|\delta) = \delta, \qquad \text{for all } \delta \in \Delta. \tag{49}$$

Second, the distribution of supplies must satisfy the consistency condition:

$$\bar{N}'(A) = \int_{\delta} \Omega(A|\delta)\, d\bar{N}(\delta), \tag{50}$$

for all Borel sets A of Δ. This ensures that the measure of trees in the set A, $\bar{N}'(A)$, is obtained by adding up the measures of trees in A found in all replicating portfolios, $\Omega(A|\delta)d\bar{N}(\delta)$. One can directly verify that the two distributions, $\bar{N}$ and $\bar{N}'$, have identical aggregate dividend:

$$\int_{\delta'} \delta' d\bar{N}'(\delta') = \int_{\delta'} \int_{\delta} \delta' d\Omega(\delta'|\delta) d\bar{N}(\delta) = \int_{\delta} \delta d\bar{N}(\delta), \tag{51}$$

where the first equality follows from (50) (or, more precisely, from Theorem 8.3 in Stokey and Lucas 1989) and the second equality follows from (49).

The result then follows almost directly from the definition. Suppose that the frictionless market equilibrium under consideration is implemented for some θ, given some distribution $\bar{N}$. Consider a change from $\bar{N}$ to $\bar{N}'$. Then all agents can replicate their portfolio holdings under $\bar{N}$ by choosing tree portfolios $N_i'(A) = \int_{\delta} \Omega(A|\delta)\, dN_i(\delta)$, where the (t, s^t) notation is suppressed for simplicity. From equation (50), these tree portfolios satisfy market clearing. The sequential budget constraints and the incentive constraints of each agent continue to hold because, in a frictionless market equilibrium, tree prices are linear in δ and because of equation (49). This shows that agents' choices remain budget and incentive feasible, and continue to satisfy market clearing. Optimality follows because, by our maintained assumption of IC-implementability, the agents' choices attain an upper bound on their maximum attainable utility: the utility they can attain under the same price system but without incentive constraints.

Appendix E. Proof of Proposition 3

Let $V(\alpha, x) \equiv \beta g^{-\gamma}\big[\pi_1(1 + k_1\alpha)^{\phi\gamma}x + \pi_2(1 - k_2\alpha)^{\phi\gamma}(1 - x)\big]$ denote the valuation function for payoffs in the simplex, $(x, 1 - x)$. Under the maintained assumption of the proposition, $\phi\gamma < 1$, the function $\alpha \mapsto V(\alpha, x)$ is strictly concave in α. Moreover, $\partial V/\partial\alpha = \phi\gamma\big(k_1\pi_1(1 + k_1\alpha)^{\phi\gamma-1}x - k_2\pi_2(1 - k_2\alpha)^{\phi\gamma-1}(1 - x)\big)$ is positive at $\alpha = 0$ if and only if

$$x \geq x^{\star}(0) = \frac{k_2\pi_2}{k_1\pi_1 + k_2\pi_2}, \tag{52}$$

and goes to minus infinity as $\alpha \to 1/k_2$. It follows that all payoffs $x \leq x^{\star}$ belong to the optimal payoff set of $\alpha_1 = 0$. For payoffs $x > x^{\star}(0)$, there exists some $\alpha^{\star}(x) > 0$ such that $V(x, \alpha)$ achieves a strict maximum at $\alpha = \alpha^{\star}(x)$. Given that $\partial^2 V/\partial x \partial\alpha > 0$, it follows that $\alpha^{\star}(x)$ is strictly increasing. One can also verify that $\alpha^{\star}(x)$ goes to $1/k_2$ as $x \to 1$. Now considering the inverse function, it follows that, for each $\alpha \in (0, 1/k_2]$, there exists some $x^{\star}(\alpha) \in \big(x^{\star}(0), 1\big]$ such that α has strictly higher valuation than any other agents for all payoffs near $x^{\star}(\alpha)$. Now since we have a finite number of agents $\alpha_1 = 0 < \alpha_2 < \cdots < \alpha_I$, it follows that, for all $i > 2$, α_i values all payoffs near $x^{\star}(\alpha_i)$ the most. Together with our earlier observations that X_i are intervals, increasing in α_i, it follows that there is a strictly increasing sequence:

$$x_0 = 0 < x_1 < x_2 < \cdots < x_I = 1,$$

such that agent i holds all the payoffs in $X(\alpha_i) = [x_{i-1}, x_i]$, and $x_{i-1} < x^\star(\alpha_i) < x_i$. ∎

Appendix F. Proof of Proposition 4

Suppressing the (t, s^t) notation for simplicity, let $\delta \mapsto x(\delta) \in \mathbb{R}_+^S$ denote the function that maps a tree δ into its vector of one-period-ahead payoffs. Let $\hat{M}$ denote the measure on payoffs $x \in \mathbb{R}_+^S$, induced by the portfolio M: that is, for all Borel set B of $\mathbb{R}_+^S$, $\hat{M}(B) = M(x^{-1}(B))$. With this notation, the replicating portfolio condition writes $x(\delta) = \int x' d\hat{M}(x')$. Let i be the type of some agent who is the best holder of tree δ, that is $x(\delta) \in X_i$. With this notation, the maintained assumption of the proposition implies that X_i does not contain (almost) all trees of the replicating portfolio, or $\hat{M}(\mathbb{R}_+^S \backslash X_i) > 0$. The price of the replicating portfolio writes

$$\int P(\delta') dM(\delta') = \int \max_j Q_j \cdot x(\delta') dM(\delta') = \int \max_j Q_j \cdot x' d\hat{M}(x')$$

$$> \int Q_i \cdot x' d\hat{M}(x') = P(\delta),$$

where the first equality on the first line follows by definition of the price function for trees; the second equality on the first line follows by the change of variable formula (see Theorem 13.46 in Aliprantis and Border 2006); the strict inequality on the second line follows because $\max_j Q_j \cdot x' > Q_i \cdot x'$ for all $x' \in \mathbb{R}_+^S \backslash X_i$ and $\hat{M}(\mathbb{R}_+^S \backslash X_i) > 0$; the equality on the second line follows because the payoff of the portfolio $\hat{M}$ replicate the payoff of tree δ, and because the price of tree δ is equal to the private valuation of agent i for that tree. ∎

Appendix G. Proof of Proposition 5

The proposition is an implication of the following Lemma.

LEMMA G.1: *Let $f(x, y)$ be some real-valued function of $(x, y) \in \mathbb{R}^K \times \mathbb{R}$. Assume that, $f(x, y)$ is convex in (x, y), and strictly increasing in y. Suppose that, for each x, $f(x, y) = 0$ has a solution, denoted by $\phi(x)$. Then, $\phi(x)$ is concave in x.*

For a proof, note that the convexity of $f(x, y)$ with respect to (x, y) implies

$$f(w_1 x_1 + w_2 x_2, w_1 \phi(x_1) + w_2 \phi(x_2)) \leq w_1 f(x_1, \phi(x_1)) + w_2 f(x_2, \phi(x_2)) = 0.$$

But since $f(x, y)$ is strictly increasing in y, $w_1 \phi(x_1) + w_2 \phi(x_2) \leq \phi(w_1 x_1 + w_2 x_2)$.

Now let us turn to the proposition. Omitting the δ and s^{t+1} arguments for notational simplicity, we first note that, in terms of returns and agent-specific stochastic discount factors, the pricing formula (26) writes

$$1 = \max_i E_t[M_{it+1} R_{t+1}].$$

Now using the factor decomposition of returns, (28), together with the assumed orthogonality condition $E_t[M_{it+1} \epsilon_{t+1}] = 0$, this can be written,

$$\max_i \left\{ E_t[R_{t+1}] E_t[M_{it+1}] + \sum_{k=1}^{K} \beta_k E_t[M_{it+1} F_{kt+1}] - 1 \right\} = 0. \tag{53}$$

The left-hand side can be written as $f(x,y) = 0$, where x is the vector of betas, $(\beta_k)_{k=1}^{K}$, and y is the expected return of the asset, $E_t[R_{t+1}]$. The function $f(x,y)$ is convex, since it is the maximum of affine functions of (x,y). It is strictly increasing in y, since the stochastic discount factors are strictly positive. We also have that $\lim_{y \to -\infty} f(x,y) = -\infty$ and $\lim_{y \to +\infty} f(x,y) = +\infty$, hence $f(x,y) = 0$ has a unique solution. Now we apply Lemma G.1 and obtain that the solution $\phi(x)$ of the equation $f(x,y) = 0$ is concave in x. ■

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