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Online Appendix to
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The proofs in this document are analytical, but computer-assisted. Each proof therefore comes with a companion Mathematica file.[1] The document is organized as follows. In Section A, we present a normalized version of the model which will be used in the remainder of this document. Proposition 1 is proven in Section B. We study the Pareto efficiency and stability properties of symmetric collusive-like equilibria in Section C. Lemma 5 is proven in Section D. Proposition 3 is proven in Section E. We show that our results are robust to assuming that unintegrated downstream firms choose their supplier before downstream prices are set in Section F. Proposition 4 is proven in Section G. Our results on two-part tariffs (Propositions 5 and 6) are proven in Section H. Our results on secret offers (Propositions 7 and 8) are proven in Section I.

A Normalization

Consider the following two models:

- Model 1 is the model defined in the paper.

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[1] Available at sites.google.com/site/nicolasschutz/research.

- Model 2 is as follows. Preferences are represented by the utility function

$$\hat{U} = \hat{q}_0 + \sum_{k=1}^{N} \hat{q}_k - \frac{1}{2}\left(\sum_{k=1}^{N} \hat{q}_k\right)^2 - \frac{N}{2(1+\hat{\gamma})}\left(\sum_{k=1}^{N} \hat{q}_k^2 - \frac{(\sum_{k=1}^{N} \hat{q}_k)^2}{N}\right),$$

  which generates the demand system

$$\hat{q}_k = \frac{1}{N}\left(1 - \hat{p}_k - \hat{\gamma}\left(\hat{p}_k - \frac{\sum_{k'=1}^{N} \hat{p}_{k'}}{N}\right)\right).$$

  The profit of a vertically integrated firm after K mergers is given by:

$$\hat{\pi}_i = \hat{p}_i \hat{q}_i + \hat{w}_i \sum_{k=K+1}^{N} 1\left[s_k = i\right] \hat{q}_k.$$

  A downstream firm purchasing the input at price $\hat{w}$ earns:

$$\hat{\pi}_k = (\hat{p}_k - \hat{w} - \hat{\delta})\hat{q}_k.$$

Let $\hat{\delta} = \delta/(1-m+\delta)$, $\hat{\gamma} = \gamma N/\mathcal{N}$, $\hat{p}_k = (p_k - m + \delta)/(1-m+\delta)$ for all k, and $\hat{w}_i = (w_i - m)/(1-m+\delta)$ for all i. It is straightforward to show that

$$\begin{aligned}
\hat{q}_k &= \frac{1}{1-m+\delta}\frac{\gamma N + \mathcal{N}}{(1+\gamma)N} \times q_k \quad \forall k \in \{1,\ldots,N\},\\
\hat{\pi}_k &= \frac{1}{(1-m+\delta)^2}\frac{\gamma N + \mathcal{N}}{(1+\gamma)N} \times \pi_k \quad \forall k \in \{1,\ldots,N\},\\
\text{and } \hat{W}(\lambda) &= \frac{1}{(1-m+\delta)^2}\frac{\gamma N + \mathcal{N}}{(1+\gamma)N} \times W(\lambda).
\end{aligned}$$

This means that model 2 is a normalized version of model 1. Throughout this document, we focus on model 2, keeping in mind that all the results we obtain also apply to model 1. From now on, we remove the hats on model 2 variables.

B Proof of Proposition 1

All calculations are in the Mathematica notebook `01_equilibrium_characterization.nb`. We decompose Proposition 1 into two lemmas. In the statements and proofs of those lemmas, we ignore the non-negativity constraint on δ. To obtain the results as they are stated in the paper, it suffices to replace the thresholds δ_m, $\underline{\delta}_c$, and $\overline{\delta}_c$ by $\max(0, \delta_m)$, $\max(0, \underline{\delta}_c)$, and $\max\left(0, \overline{\delta}_c\right)$.

Lemma A. *There exists a threshold $\delta_m(M, N, \gamma)$ such that monopoly-like equilibria exist if and only if $\delta \geq \delta_m(M, N, \gamma)$.*

Proof. We start by computing equilibrium prices and profits (Step 001 in the Mathematica notebook). We also define two thresholds: δ_{sup} and w_{max} (Step 002). δ_{sup} is the threshold above which downstream firms cannot be active when $w = 0$. When $\delta < \delta_{sup}$, w_{max} is the upstream price threshold above which downstream firms cannot be active. In the following, we assume $\delta < \delta_{sup}$. $\Pi(w, 1)$ is concave in w and we define w_m as the monopoly upstream price (Step 003). We also define δ^m_{sup} as the threshold above which the monopoly upstream price is no longer interior. In the following, we assume that $\delta < \delta^m_{sup}$, in line with footnote 18 of the paper.

$\Pi(w_m, 0) - \Pi(w_m, 1)$ is concave in δ and positive for $\delta = \delta_{sup}$ (Step 004). Therefore, there exists a unique $\delta_m < \delta_{sup}$ such that $\Pi(w_m, 0) - \Pi(w_m, 1) \geq 0$ if and only if $\delta \geq \delta_m$. □

Lemma B. *There exist two thresholds $\underline{\delta}_c(M, N, \gamma)$ and $\overline{\delta}_c(M, N, \gamma)$, where $\underline{\delta}_c(M, N, \gamma) < \delta_m(M, N, \gamma) < \overline{\delta}_c(M, N, \gamma)$, such that collusive-like equilibria exist if and only if $\underline{\delta}_c(M, N, \gamma) \leq \delta < \overline{\delta}_c(M, N, \gamma)$. When this condition is satisfied, the set of input prices that can be sustained in a symmetric collusive-like equilibrium is an interval.*

Proof. $\Pi(w, 1/M) - \Pi(w, 1)$ is convex in w, hence positive outside its roots, 0 and w_1 (Step 005). $\Pi(w, 1/M) - \Pi(w, 0)$ is concave in w, hence positive between its roots, 0 and w_2 (Step 006). There exists δ_1 such that $0 < w_1 < w_2$ if $\delta < \delta_1$, and $w_2 \leq w_1 \leq 0$ if $\delta \geq \delta_1$ (Step 007). It follows that any symmetric collusive-like equilibrium price must be between w_1 and w_2, and that if $\delta \geq \delta_1$, then there are no collusive-like equilibria.

In the following, we assume $\delta < \delta_1$. We have $w_2 < w_{max}$ (Step 008). There exists $\delta_2 < \delta_1$ such that $w_2 \leq w_m$ if and only if $\delta_2 \leq \delta \leq \delta_1$ (Step 009). Therefore, when $\delta \in [\delta_2, \delta_1]$, there exists a continuum of collusive-like equilibria between w_1 and w_2.

There exists $\delta_3 < \delta_2$ such that $w_1 \leq w_m$ if and only if $\delta_3 \leq \delta \leq \delta_2$ (Step 010). $\Pi(w, 1/M)$ is concave in w and reaches its maximum at w_c (Step 011). Moreover, $w_c > w_2$ for $\delta \geq \delta_3$ (Step 012). If $\delta_3 \leq \delta \leq \delta_2$, then $w_1 \leq w_m \leq w_2 \leq w_c$. For all $w \in [w_1, w_m]$, $\Pi(w, 1) \leq \Pi(w, 1/M)$, and so there is a continuum of collusive-like equilibria between w_1 and w_m. For all $w \in [w_m, w_2]$,

$$\Pi\left(w, \frac{1}{M}\right) \geq \Pi\left(w_m, \frac{1}{M}\right) \geq \Pi(w_m, 1),$$

and so there is a continuum of collusive-like equilibria between w_m and w_2. We can conclude that if $\delta \in [\delta_3, \delta_1]$, then there exists a continuum of collusive-like equilibria between w_1 and w_2.

Next, assume $\delta < \delta_3$. Then, $w_m < w_1 < w_2$. There exists $\delta_4 < \delta_3$ such that $w_2 > w_c$ if and only if $\delta < \delta_4$ (Step 013). There exists $\delta_5 < \delta_4$ such that $w_1 > w_c$ if and only if $\delta < \delta_5$ (Step 014).

Suppose first that $\delta < \delta_5$. Then, $\max(w_m, w_c) < w_1 < w_2$. Hence, for every $w \in (w_1, w_2)$,

$$\Pi\left(w, \frac{1}{M}\right) < \Pi\left(w_1, \frac{1}{M}\right) = \Pi(w_1, 1) < \Pi(w_m, 1).$$

Therefore, there are no collusive-like equilibria.

Next, assume $\delta_5 < \delta < \delta_4$. Then, $w_m < w_1 < w_c < w_2$. The best candidate for a collusive-like equilibrium is therefore $w = w_c$. The function $d\pi_c(\delta) \equiv \Pi(w_c, 1/M) - \Pi(w_m, 1)$ is concave in δ (Step 015), positive when $\delta = \delta_3$, and negative when $\delta = \delta_5$.[2] There are two cases to distinguish:

1. If $d\pi_c(\delta_4) \leq 0$, then by concavity and since $d\pi_c(\delta_3) > 0$, we have that $d\pi_c(\delta) < 0$ for all $\delta_5 < \delta < \delta_4$. In this case, there are no collusive-like equilibria.

2. Conversely, if $d\pi_c(\delta_4) > 0$, then, there exists a unique threshold $\delta_6 \in (\delta_5, \delta_4)$ such that $d\pi_c(\delta) \geq 0$ if and only if $\delta \in [\delta_6, \delta_4]$. When this condition holds, there is a continuum of collusive-like equilibria between $\tilde{w}_1$ and $\tilde{w}_2$, where $\tilde{w}_1$ and $\tilde{w}_2$ are the unique solutions of equation $\Pi(w, 1/M) = \Pi(w_m, 1)$ on the intervals $[w_1, w_c]$ and $[w_c, w_2]$, respectively. Otherwise, there are no collusive-like equilibria.

Finally, assume $\delta_4 < \delta < \delta_3$. Then, $w_m < w_1 < w_2 < w_c$. The best candidate for a collusive-like equilibrium is therefore $w = w_2$. The function $d\pi_2(\delta) \equiv \Pi(w_2, 1/M) - \Pi(w_m, 1)$ is quadratic in δ (Step 016), positive when $\delta = \delta_3$ and negative when $\delta = \delta_5$. Therefore, there exists a unique $\delta_7 \in (\delta_5, \delta_3)$ such that $d\pi_2(\delta_7) = 0$. Note in addition that $d\pi_2(\delta_4) = d\pi_c(\delta_4)$. Again, we have to consider two cases:

1. If $d\pi_2(\delta_4) = d\pi_c(\delta_4) < 0$, then $\delta_7 > \delta_4$. Therefore, if $\delta \in [\delta_4, \delta_7)$, then there are no collusive-like equilibria. If $\delta \geq \delta_7$, then there is a continuum of collusive-like equilibria between $\tilde{w}_1$ and w_2, where $\tilde{w}_1$ is the unique solution of equation $\Pi(w_m, 1) = \Pi(w, 1/M)$ on the interval (w_1, w_2).

[2] If $\delta = \delta_3$, then $w_1 = w_m < w_2 < w_c$, implying that

$$\Pi\left(w_c, \frac{1}{M}\right) > \Pi\left(w_1, \frac{1}{M}\right) = \Pi\left(w_m, \frac{1}{M}\right).$$

If $\delta = \delta_5$, then $w_m < w_1 = w_c$, implying that $\Pi(w_c, 1/M) < \Pi(w_m, 1)$.

2. If $d\pi_2(\delta_4) = d\pi_c(\delta_4) \geq 0$, then $\delta_7 \leq \delta_4$. Therefore, for all $\delta \in [\delta_4, \delta_3]$, there is a continuum of collusive- like equilibria between $\tilde{w}_1$ and w_2, where $\tilde{w}_1$ is the unique solution of equation $\Pi(w_m, 1) = \Pi(w, 1/M)$ on interval (w_1, w_2).

Defining $\underline{\delta}_c$ as

$$\underline{\delta}_c = \begin{cases} \delta_6 & \text{if } d\pi_c(\delta_4) > 0, \\ \delta_7 & \text{if } d\pi_c(\delta_4) \leq 0, \end{cases}$$

and $\overline{\delta}_c = \delta_1$ and checking that $\max(\delta_6, \delta_7) < \delta_3 < \delta_m$ (Step 017) and $\delta_1 > \delta_m$ (Step 018) concludes the proof of the lemma. □

Combining Lemmas A and B proves Proposition 1.

C Properties of Symmetric Collusive-Like Equilibria

C.1 Stability

Throughout this subsection, we assume that a symmetric distribution of upstream market shares is selected in stage 3 whenever multiple vertically integrated firms offer the same input price. We show that any interior symmetric collusive-like equilibrium is stable when $M = 2$, and provide conditions under which symmetric collusive-like equilibria are stable when $M \geq 3$.

$M = 2$**.** We say that an equilibrium of the upstream competition subgame is stable if, following a small perturbation of upstream prices and letting upstream firms myopically and sequentially best-respond to each other until a new equilibrium is reached, upstream prices in the new equilibrium are close to those in the initial equilibrium. To state this definition of stability formally, we let $BR_i(w_j)$ denote vertically integrated firm $U_i - D_i$'s best-response in upstream price to $U_j - D_j$'s upstream price w_j, $j \neq i$. We have $BR_i(w_j) = \arg\max_{w_i} \widetilde{\Pi}_i(w_i, w_j)$, where

$$\widetilde{\Pi}_i(w_i, w_j) = \begin{cases} \Pi(w_i, 1) & \text{if } w_i < w_j, \\ \Pi(w_i, 1/2) & \text{if } w_i = w_j, \\ \Pi(w_j, 0) & \text{if } w_i > w_j. \end{cases}$$

Definition 1. *An equilibrium with upstream prices* (w_1, w_2) *is* stable *if, for any small* $\varepsilon > 0$*, there exists* $\eta > 0$ *such that for any sequence* $(w_1^n, w_2^n)_{n\geq 0}$ *that satisfies*

- $(w_1^0, w_2^0) \in [w_1 - \eta, w_1 + \eta] \times [w_2 - \eta, w_2 + \eta]$,
- $w_1^{n+1} \in BR_1(w_2^n)$ *and* $w_2^{n+1} = w_2^n$ *for even* $n \geq 0$,
- *and* $w_1^{n+1} = w_1^n$ *and* $w_2^{n+1} \in BR_2(w_1^n)$ *for odd* $n \geq 0$,

there exists $(w_1^\infty, w_2^\infty) \in [w_1 - \varepsilon, w_1 + \varepsilon] \times [w_2 - \epsilon, w_2 + \epsilon]$ *such that* $(w_1^n, w_2^n)_{n \geq 0}$ *converges to* (w_1^∞, w_2^∞).

Recall from Proposition 1 that when a collusive-like equilibrium exists, the set of prices that can be sustained in a symmetric collusive-like equilibrium is an interval. Denote this interval by $[\underline{w}_c, \overline{w}_c]$. We have:

Proposition A. *Assume* $M = 2$ *and* $\delta \in (\underline{\delta}_c, \overline{\delta}_c)$. *Any symmetric collusive-like equilibrium with price* $w \in (\underline{w}_c, \overline{w}_c)$ *is stable.*

Proof. Consider a symmetric collusive-like equilibrium at price $w \in (\underline{w}_c, \overline{w}_c)$. Let $\varepsilon > 0$ such that $[w - \varepsilon, w + \varepsilon] \subset (\underline{w}_c, \overline{w}_c)$, and consider a perturbation $(w_1^0, w_2^0) \in [w - \varepsilon, w + \varepsilon]^2$. Since $w_2^0 \in (\underline{w}_c, \overline{w}_c)$, w_2^0 can be sustained in a symmetric collusive-like equilibrium and the equilibrium condition (8) holds strictly: $\Pi(w, 1/2) > \max\{\max_{\widetilde{w} \leq w} \Pi(\widetilde{w}, 1), \Pi(w, 0)\}$ at $w = w_2^0$. (See the proof of Proposition 1.) Thus, $U_1 - D_1$'s unique best-response to w_2^0 is $BR_1(w_2^0) = w_2^0$. The new equilibrium is reached in one step and is a symmetric collusive-like equilibrium at price w_2^0, which is within distance ε from the initial equilibrium. □

$M \geq 3$**.** Definition 1 can easily be adapted to allow for more than 2 vertically integrated firms: For every $1 \leq i \leq M$, $U_i - D_i$ best-responds in periods $n \geq 1$ such that $i = n$ modulo M, and only in those periods. The best-response function BR_i should be redefined as $BR_i(w_{-i}) = \arg\max_{w_i} \widetilde{\Pi}_i(w_i, w_{-i})$, where

$$\widetilde{\Pi}_i(w_i, w_{-i}) \equiv \begin{cases} \Pi\left(w_i, \frac{1}{|\{1 \leq j \leq M:\ w_j = w_i\}|}\right) & \text{if } w_i = \min_{1 \leq k \leq M} w_k, \\ \Pi\left(\min_{1 \leq k \leq M} w_k, 0\right) & \text{otherwise.} \end{cases}$$

We have:

Proposition B. *Assume* $M \geq 3$. *For every* $\delta \in [\delta_m, \overline{\delta}_c)$, *there exists a non-empty interval of input prices* $(\underline{w}, \overline{w})$ *such that for every* $w \in (\underline{w}, \overline{w})$, *the symmetric collusive-like equilibrium at price* w *is stable.*

Proof. In Step 019 of the Mathematica notebook `01_equilibrium_characterization.nb`, we show the following: For every $\delta \in [\delta_m, \overline{\delta}_c)$, there exists a unique $w_* > 0$ such that $\Pi(w_*, 1) = \Pi(w_*, 0)$; moreover, $w_* \leq w_m$.

By strict concavity of $\Pi(w, \cdot)$, we have that for every $\alpha \in \{1/M, 1/(M-1), \dots, 1/2\}$,

$$\begin{aligned}\Pi(w_*, \alpha) &> \alpha\Pi(w_*, 1) + (1-\alpha)\Pi(w_*, 0) \\ &= \Pi(w_*, 1) = \Pi(w_*, 0) \\ &= \max\left(\Pi(w_*, 0), \max_{\tilde{w} \leq w_*} \Pi(\tilde{w}, 1)\right),\end{aligned}$$

where the last line follows from the fact that $w_* \leq w_m$. By continuity of $\Pi(\cdot, \alpha)$, this implies the existence of $\underline{w} < w_* < \overline{w}$ such that for every $\alpha \in \{1/M, 1/(M-1), \dots, 1/2\}$ and $w \in (\underline{w}, \overline{w})$,

$$\Pi(w, \alpha) > \max\left(\Pi(w, 0), \max_{\tilde{w} \leq w} \Pi(\tilde{w}, 1)\right). \tag{1}$$

Let $w \in (\underline{w}, \overline{w})$, $\varepsilon > 0$ such that $[w-\varepsilon, w+\varepsilon] \subseteq (\underline{w}, \overline{w})$, and $(w_j^0)_{1 \leq j \leq M} \in [w-\varepsilon, w+\varepsilon]^M$. By inequality (1), $U_1 - D_1$'s unique best response to w_{-1}^0 is to set $w_1^1 = \hat{w} \equiv \min_{2 \leq j \leq M} w_j^0$. The inequality also implies that $U_2 - D_2$'s unique best response to $(\hat{w}, w_3^0, \dots, w_M^0)$ is to set $w_2^2 = \hat{w}$. Continuing this process up to $U_M - D_M$, we obtain a new profile of input prices in which each vertically integrated firm sets $\hat{w}$. Moreover, by inequality (1), input prices no longer change after the M-th step has been reached. We have thus converged to a symmetric collusive-like equilibrium at price $\hat{w} \in [w-\varepsilon, w+\varepsilon]^M$. It follows that the symmetric collusive-like equilibrium at price w is stable. □

C.2 Pareto efficiency

Define $\widehat{w}_c \equiv \arg\max_{w \in [\underline{w}_c, \overline{w}_c]} \Pi(w, 1/M)$ as the input price such that, among symmetric collusive-like equilibria, the one at price $\widehat{w}_c$ generates the highest profits for the vertically integrated firms.

Proposition C. *If $M > 2$ or $N > 3$ or γ is low enough, then there exists $\widehat{\delta}_c > \delta_m$ such that for every $\delta \in [\underline{\delta}_c, \widehat{\delta}_c]$, the symmetric collusive-like equilibrium at price $\widehat{w}_c$ is a Pareto optimum from the viewpoint of the vertically integrated firms.*

Proof. The condition that $M > 2$ or $N > 3$ or γ is low enough in the statement of the proposition guarantees that the thresholds δ_m is strictly positive (see Step 020 in the Mathematica notebook `01_equilibrium_characterization.nb`).

In the symmetric collusive-like equilibrium at price $\widehat{w}_c$, every vertically integrated firm earns $\Pi(\widehat{w}_c, 1/M)$. We need to show that this equilibrium is not Pareto-dominated by another collusive-like equilibrium or by the monopoly-like equilibrium. Consider another collusive-like equilibrium at price w with some distribution of upstream market shares $(\alpha_j)_{1 \leq i \leq M}$. Suppose $w \neq \widehat{w}_c$ or $(\alpha_j)_{1 \leq j \leq M} \neq (1/M, \ldots, 1/M)$, and let

$$i \in \arg \min_{1 \leq j \leq M} \Pi(w, \alpha_j).$$

Then, by concavity of $\Pi(w, \cdot)$ and by the definition of $\widehat{w}_c$, we have that

$$\Pi(w, \alpha_i) \leq \Pi\left(w, \frac{1}{M}\right) \leq \Pi\left(\widehat{w}_c, \frac{1}{M}\right).$$

Since $\Pi(w, \cdot)$ is strictly concave and $w \neq \widehat{w}_c$ or $(\alpha_j)_{1 \leq j \leq M} \neq (1/M, \ldots, 1/M)$, at least one of these inequalities is strict, implying that the collusive-like equilibrium with price w and market shares $(\alpha_j)_{1 \leq j \leq M}$ does not Pareto dominate the symmetric collusive-like equilibrium at price $\widehat{w}_c$. This implies in particular that the symmetric collusive-like equilibrium at $\widehat{w}_c$ is a Pareto optimum from the viewpoint of the vertically integrated firms if $\delta \in [\underline{\delta}_c, \delta_m)$.

Next, consider the monopoly-like equilibrium, which exists if and only if $\delta \geq \delta_m$. When $\delta = \delta_m$, the existence condition (7) holds with equality, $\Pi(w_m, 1) = \Pi(w_m, 0)$. Since $\Pi(w, \alpha)$ is strictly concave in α, we have $\Pi(w_m, 1/M) > \min\{\Pi(w_m, 1), \Pi(w_m, 0)\}$. Hence, in the monopoly-like equilibrium, vertically integrated firms earn strictly less than $\Pi(w_m, 1/M)$, which is less than $\Pi(\widehat{w}_c, 1/M)$ by definition of $\widehat{w}_c$. By continuity, this remains true when δ is in the right neighborhood of δ_m, which proves the existence of $\widehat{\delta}_c > \delta_m$. □

D Proof of Lemma 5

Proof. Mathematical derivations for this proof can be found in the Mathematica notebook `02_first_mergers_welfare.nb`. We begin by computing equilibrium prices, quantities, and profits as a function of K, the number of vertical mergers that have occurred, assuming throughout that the input is priced at marginal cost (Step 001). Next, we define $\overline{\delta}(K)$ as the cutoff synergy level above which unintegrated downstream firms can no longer be active after K mergers (Step 002). There we also show that $\overline{\delta}(K)$ is strictly decreasing in K. In the following, we assume that $\delta < \overline{\delta}(M) \equiv \overline{\delta}$.

Next, we compute equilibrium consumer surplus, producer surplus, and aggregate surplus (Step 003). Treating K as a continuous variable, we show that consumer surplus is strictly concave in K, and that the derivative of consumer surplus with respect to K evaluated at $K = M$ is strictly positive (Step 004). It follows that consumer surplus is strictly increasing in K. A similar argument implies that aggregate surplus is strictly increasing in K (Step 005). Since our market performance measure is a convex combination of consumer surplus and aggregate surplus, it follows that market performance is strictly increasing in K. □

E Proof of Proposition 3

We split the proposition into two lemmas, which provide a more precise statement of our results.

Lemma C. *Consider the case where $M = 2$ and $N = 3$. Under selection criterion ($\mathcal{C}$), there exist functions $\gamma_c : [0, 1] \longrightarrow \mathbb{R}_{++}$ and*

$$\delta_c^W : \{(\gamma, \lambda) \in \mathbb{R}_{++} \times [0, 1] : \ \gamma > \gamma_c(\lambda)\} \longrightarrow \mathbb{R}_+$$

such that for every $(\delta, \gamma, \lambda)$, the second merger worsens market performance if and only if $\gamma > \gamma_c(\lambda)$ and $\delta \in \big[\underline{\delta}_c(\gamma), \delta_c^W(\gamma)\big)$.

Moreover, γ_c is strictly increasing in λ, δ_c^W is strictly decreasing in both of its arguments and satisfies $\underline{\delta}_c(\gamma) < \delta_c^W(\gamma, \lambda) < \overline{\delta}_c(\gamma)$, and $\underline{\delta}_c(\gamma)$ is strictly decreasing on some interval $(0, \widehat{\gamma}_c]$, and identically equal to zero on $[\widehat{\gamma}_c, \infty)$.

Proof. All calculations are in the Mathematica notebook `03_welfare_criterion_c.nb`. We begin by computing equilibrium prices, consumer surplus, and producer surplus in the one-merger subgame (Step 001). Turning our attention to the two-merger subgame, we recompute equilibrium prices and profits at the equilibrium of stage 3 as a function of w and (α_1, α_2) (Step 002). Having done that, we can recalculate the thresholds w_m, w_1, w_2, w_c, $\delta_1 (= \overline{\delta}_c)$, δ_3, and δ_4, which were defined and used in the proof of Proposition 1 (Step 003). There, we also redefine $d\pi_2(\delta) \equiv \Pi(w, 1/2) - \Pi(w_m, 1)$. We show that $d\pi_2(\delta_4) < 0$ and $d\pi_2''(\delta) < 0$, which, as shown in the proof of Proposition 1, implies that $\underline{\delta}_c$ is equal to the smallest root of $d\pi_2(\cdot)$ (Step 004). Moreover, the argument in the proof of Proposition 1 implies that the symmetric collusive-like equilibrium that maximizes the vertically integrated firms' profits is the one with input price w_2. Next, we compute producer surplus and consumer surplus in that symmetric collusive-like equilibrium (Step 005). We also define the merged-induced change in market performance as $f(\delta, \gamma, \lambda)$.

The next step is to study the behavior of the function f. We show that f is strictly concave in δ, $f(\overline{\delta}_c(\gamma), \gamma, \lambda) > 0$, and $f(0, \gamma, \lambda) < 0$ (Step 006). For a given $\gamma > 0$, those properties have the following implications: If $f(\underline{\delta}_c(\gamma), \gamma, \lambda) \geq 0$, then $f(\delta, \gamma, \lambda) > 0$ for every $\gamma \in (\underline{\delta}_c(\gamma), \overline{\delta}_c(\gamma))$; if instead $f(\underline{\delta}_c(\gamma), \gamma, \lambda) < 0$, then there exists $\delta_c^W(\gamma, \lambda) \in (\underline{\delta}_c(\gamma), \overline{\delta}_c(\gamma))$ such that for every $\delta \in (\underline{\delta}_c(\gamma), \overline{\delta}_c(\gamma))$, $f(\delta, \gamma, \lambda) < 0$ if $\delta < \delta_c^W(\gamma, \lambda)$ and $f(\delta, \gamma, \lambda) > 0$ if $\delta > \delta_c^W(\gamma, \lambda)$. Moreover, whenever $\delta_c^W(\gamma, \lambda)$ is well defined, it is the smallest root of the quadratic polynomial $f(\cdot, \gamma, \lambda)$.

Next, we show that $\underline{\delta}_c(\gamma) > 0$ if and only if $\gamma < \gamma_c^2 \simeq 7.49$ (Step 007). The argument in the previous paragraph implies that $\delta_c^W(\gamma, \lambda)$ is well defined whenever $\gamma \geq \gamma_c^2$. In the following, we assume that $\gamma < \gamma_c^2$ and study the sign of $\phi(\gamma, \lambda) \equiv f(\underline{\delta}_c(\gamma), \gamma, \lambda)$.

The function ϕ is linear and strictly increasing in λ (Step 008). Hence, there exists a unique $\Lambda(\gamma) \in \mathbb{R}$ such that $\phi(\gamma, \Lambda(\gamma)) = 0$. Moreover, there exist two cutoffs, γ_c^1 and $\gamma_c^{1\prime}$ such that

$$2.75 \simeq \gamma_c^1 < \gamma_c^{1\prime} \simeq 4.83,$$

$\Lambda(\gamma) < 0$ if and only if $\gamma < \gamma_c^1$, and $\Lambda(\gamma) > 1$ if and only if $\gamma > \gamma_c^{1\prime}$ (Step 009). The monotonicity properties of ϕ in λ imply that $\phi(\gamma, \lambda) \geq 0$ for every $\lambda \in [0, 1]$ and $\gamma \in (0, \gamma_c^1]$, and $\phi(\gamma, \lambda) < 0$ for every $\lambda \in [0, 1]$ and $\gamma \in (\gamma_c^{1\prime}, \gamma_c^2)$. Hence, $f(\delta, \gamma, \lambda) > 0$ for every $\gamma < \gamma_c^1$, $\delta \geq \underline{\delta}_c(\gamma)$, and $\lambda \in [0, 1]$, whereas $\delta_c^W(\gamma, \lambda)$ is well defined for every $\gamma > \gamma_c^{1\prime}$ and $\lambda \in [0, 1]$.

Next, we assume $\gamma \in [\gamma_c^1, \gamma_c^{1\prime}]$ and investigate whether $\delta_c^W(\gamma, \lambda)$ is well defined. The restriction of $\Lambda(\cdot)$ to the domain $[\gamma_c^1, \gamma_c^{1\prime}]$ has a strictly positive derivative (Step 010). Hence, its inverse function, $\Gamma(\cdot)$, is well defined, strictly increasing, and maps $[0, 1]$ one-to-one onto $[\gamma_c^1, \gamma_c^{1\prime}]$. It follows that, for every $\lambda \in [0, 1]$, $\delta_c^W(\cdot, \lambda)$ is well defined on $(\Gamma(\lambda), \infty)$, and $f(\delta, \gamma, \lambda) \geq 0$ whenever $\gamma \leq \Gamma(\lambda)$.

Finally, we show that the function δ_c^W has the monotonicity properties asserted in the statement of the proposition. We compute δ_c^W in closed form (Step 011). We then show that $\partial\delta_c^W/\partial\lambda$ and $\partial\delta_c^W/\partial\gamma$ are both strictly negative (Step 012). We also show that $\partial\underline{\delta}_c/\partial\gamma < 0$ whenever $\underline{\delta}_c$ is strictly positive (Step 013).

Setting $\widehat{\gamma}_c = \gamma_c^2$ and $\gamma_c = \Gamma$ concludes the proof. □

Lemma D. *Consider the case where $M = 2$ and $N = 3$. Under selection criterion ($\mathcal{M}$), there exist functions*

$$\gamma_m^1 : [0, 1] \longrightarrow (0, \infty), \quad \gamma_m^2 : [0, 1] \longrightarrow (0, \infty],$$

and

$$\delta_m^W : \left\{(\gamma, \lambda) \in \mathbb{R}_{++} \times [0, 1] : \ \gamma_m^1(\lambda) < \gamma < \gamma_2(\lambda)\right\} \longrightarrow \mathbb{R}_+$$

such that for every (δ,γ,λ), the second merger worsens market performance if and only if

- $\gamma \in (\gamma_m^1(\lambda), \gamma_m^2(\lambda))$ *and* $\delta \in [\delta_m(\gamma), \delta_m^W(\gamma,\lambda))$,
- *or* $\gamma \geq \gamma_m^2(\lambda)$ *and* $\delta \geq \delta_m(\gamma)$.

Moreover, γ_m^1 *is strictly increasing,* γ_m^2 *is increasing (and strictly so whenever it is finite),* $\gamma_1 < \gamma_2$*, and* δ_m^W *is strictly decreasing in both of its arguments and satisfies* $\delta_m(\gamma) < \delta_m^W(\gamma,\lambda) < \delta_{sup}(\gamma)$*, where* $\delta_{sup}(\gamma)$ *is the threshold above which the monopoly-like equilibrium is no longer interior.*[3] *Finally, there exists* $\widehat{\gamma}_m \in (\gamma_m^1(1), \infty)$ *such* $\delta_m(\gamma)$ *is strictly decreasing on* $(0,\widehat{\gamma}_m)$ *and identically equal to zero on* $[\widehat{\gamma}_m, \infty)$.

Proof. All calculations are in the Mathematica notebook `04_welfare_criterion_m.nb`. Consider first the two-merger subgame. We recalculate prices, quantities and profits at the equilibrium of stage 3 (Step 001). Next, we redefine w_m and δ_{sup} (Step 002). We then calculate $\hat{\delta}$, which is the δ above which $\Pi(w_m,1) \leq \Pi(w_m,0)$ (Step 003). $\hat{\delta}$ is strictly positive if and only if $\gamma < \overline{\gamma}$; it is strictly decreasing in γ for all $\gamma < \overline{\gamma}$ (Step 004). Therefore, δ_m is equal to $\hat{\delta}$ when $\gamma < \bar{\gamma}$, and to 0 when $\gamma \geq \bar{\gamma}$.

Next, we compute $W(\lambda)$ in a monopoly-like equilibrium in the two-merger game (Step 005) and in the Bertrand equilibrium in the one-merger subgame (Step 006). We define $f(\delta,\gamma,\lambda)$ as the variation in $W(\lambda)$ induced by the second merger (Step 007).

The function f is strictly increasing in δ and strictly negative when $\delta = 0$ (Step 008). Moreover, $f(\delta_{sup}(\gamma),\gamma,\lambda)$ is increasing in λ, $f(\delta_{sup}(\gamma),\gamma,1) > 0$ for all γ, and $f(\delta_{sup}(\gamma),\gamma,0) \leq 0$ if and only if $\gamma \geq \gamma_2(0)$, where $\gamma_2(0) > 0$ (Step 009). Therefore, when $\gamma < \gamma_2(0)$, $f(\delta_{sup}(\gamma),\gamma,\lambda) > 0$ for all λ. When $\gamma \geq \gamma_2(0)$, there exists a unique $\lambda_2(\gamma) \in [0,1)$ such that $f(\delta_{sup}(\gamma),\gamma,\lambda) > 0$ if and only if $\lambda > \lambda_2(\gamma)$. We show that $\lambda_2(\cdot)$ is strictly increasing on $(\gamma_2(0),\infty)$, and that $\lim_{\gamma\to+\infty} \lambda_2(\gamma) \equiv \overline{\lambda} \in (0,1)$ (Step 010). Therefore, $\lambda_2(\cdot)$ establishes a bijection from $[\gamma_2(0),\infty)$ to $[0,\overline{\lambda})$, and its inverse function, $\gamma_2(\lambda)$, is well defined and strictly increasing on the interval $[0,\overline{\lambda})$. We extend $\gamma_2(.)$ to the rest of the interval, by setting $\gamma_2(\lambda) \equiv +\infty$ for all $\lambda \geq \overline{\lambda}$.

Next, we study the sign of $f(\delta_m(\gamma),\gamma,\lambda)$. When $\gamma \geq \bar{\gamma}$, $\delta_m(\gamma) = 0$ and $f(\hat{\delta}(\gamma),\gamma,\lambda) < 0$ for all λ. Next, assume $\gamma < \bar{\gamma}$, so that $\delta_m(\gamma) = \hat{\delta}(\gamma)$. $f(\hat{\delta}(\gamma),\gamma,\lambda)$ is increasing in λ, $f(\hat{\delta}(\gamma),\gamma,0) < 0$ if and only if γ is larger than some threshold $\gamma_1(0)$, and $f(\hat{\delta}(\gamma),\gamma,1) < 0$ if and only if γ is larger than some threshold $\gamma_1(1)$, where $\gamma_1(0) < \gamma_1(1) < \bar{\gamma}$ (Step 011). Hence, if $\gamma < \gamma_1(0)$, then $f(\delta_m(\gamma),\gamma,\lambda) > 0$ for all λ. If instead $\gamma \in (\gamma_1(1),\bar{\gamma})$, then

[3] Recall from footnote 18 in the paper that the case $\delta \geq \delta_{sup}$ is ruled out by assumption.

$f(\delta_m(\gamma), \gamma, \lambda) < 0$ for all λ. Finally, if $\gamma \in (\gamma_1(0), \gamma_1(1))$, then there exists a unique $\lambda_1(\gamma) \in (0, 1)$ such that $f(\delta_m(\gamma), \gamma, \lambda) > 0$ if and only if $\lambda > \lambda_1(\gamma)$. We show that $\lambda_1(\gamma)$ is strictly increasing on the interval $[\gamma_1(0), \gamma_1(1)]$ (Step 012). Therefore, $\lambda_1(\gamma)$ establishes a bijection from $[\gamma_1(0), \gamma_1(1)]$ to $[0, 1]$, and its inverse function, $\gamma_1(\lambda)$, is well defined and strictly increasing on $[0, 1]$.

We can now conclude the welfare analysis, using the fact that f is increasing in δ:

- If $\gamma \in (0, \gamma_1(\lambda))$, then, $f(\delta, \gamma, \lambda) > 0$ for all $\delta \in [\delta_m(\gamma), \delta_{sup}(\gamma))$. It follows that the second vertical merger always improves market performance.
- If $\gamma \in (\gamma_1(\lambda), \gamma_2(\lambda))$, then, there exists a unique $\delta_m^W(\gamma, \lambda) \in (\delta_m(\gamma), \delta_{sup}(\gamma))$ such that the second merger worsens market performance if and only if $\delta \in [\delta_m(\gamma), \delta_m^W(\gamma, \lambda))$.
- If $\gamma > \gamma_2(\lambda)$, then the second merger worsens market performance if and only if $\delta \geq \delta_m(\gamma)$.

We now prove that $\delta_m^W(\gamma, \lambda)$ is strictly decreasing in both of its arguments. Assume $\gamma \in (\gamma_1(\lambda), \gamma_2(\lambda))$, so that $\delta_m^W(\gamma, \lambda)$ is well defined. By the implicit function theorem,

$$\frac{\partial \delta_W}{\partial \gamma}(\gamma, \lambda) = - \left. \frac{\partial f / \partial \gamma}{\partial f / \partial \delta} \right|_{(\delta_m^W(\gamma,\lambda),\gamma,\lambda)}.$$

We already know that $\frac{\partial f}{\partial \delta} > 0$. Moreover, we have $\frac{\partial f}{\partial \gamma}(\delta, \gamma, \lambda) > 0$ for all $\delta \in [\hat{\delta}(\gamma), \delta_{max}(\gamma)]$ (Step 013). Hence, $\partial \delta_m^W / \partial \gamma < 0$. Similarly, we prove that $\partial \delta_m^W / \partial \lambda < 0$ (Step 014).

Setting $\widehat{\gamma}_m \equiv \overline{\gamma}$, $\gamma_1^m \equiv \gamma_1$, and $\gamma_2^m \equiv \gamma_2$ concludes the proof. $\square$

F Sequential Timing

In this section, we show that the results derived in Section III of the paper continue to obtain if downstream firms choose their input supplier before downstream prices are set. Specifically, we assume that unintegrated downstream firms make publicly-observable upstream supplier choices (in stage 2.5) after upstream prices have been set (in stage 2) but before downstream competition takes place (in stage 3). We also assume that unintegrated downstream firms have access to a public randomization device: They observe the realization of a random variable θ between stages 2 and 2.5.

We extend Lemmas 2 and 3 in Section F.1 and Proposition 1 in Section F.2. Combining those results extends Proposition 2. All the calculations for this section are in the Mathematica notebook `05_timing.nb`.

F.1 Lemmas 2 and 3 under sequential timing

We have:

Lemma E. *Under sequential timing, after $K = 0, 1, \ldots, M$ mergers have taken place, the Bertrand outcome is always an equilibrium.*

Proof. Consider the K-merger subgame and suppose all upstream firms set $w_i = 0$. Assume that $U_i - D_i$ deviates upward, and that D_k (and only D_k) accepts this deviation. The resulting equilibrium downstream prices solve the following system of equations:

$$\begin{array}{lcrcl} U_i - D_i & : & \frac{\gamma}{N} w_i + 1 + \gamma \bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N}) p_i, \\ U_j - D_j,\ j \neq i & : & 1 + \gamma \bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N}) p_j, \\ D_k & : & (w_i + \delta)(1 + \gamma(1 - \frac{1}{N})) + 1 + \gamma \bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N}) p_k, \\ D_l,\ l \neq k & : & \delta(1 + \gamma(1 - \frac{1}{N})) + 1 + \gamma \bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N}) p_l. \end{array}$$

We calculate those equilibrium prices, as well as D_k's equilibrium output and profit (Step 001). We show that the derivative of D_k's equilibrium profit with respect to w_i has the opposite sign to D_k's output (Step 002), which implies that D_k's profit is strictly decreasing in w_i. It follows that D_k's profit goes down when it accepts the deviation. Therefore, there exists an equilibrium in which no downstream firm accepts $U_i - D_i$'s offer, which makes this deviation non-profitable. We then extend this result to the case in which it is an unintegrated upstream firm that deviates upward (step 003). (We do not examine downward deviations since pricing below cost is not allowed in the upstream market. It is easily shown that such deviations would not be profitable either.) □

Thus, the Bertrand outcome is always an equilibrium. It is easily shown that it is the unique equilibrium in the zero-merger subgame. In subgames with $K \in \{1, \ldots, M-1\}$ mergers, there may exist equilibria in which the input is priced above cost. The intuition for why such equilibria may exist is again related to the tradeoff between the upstream profit effect and the softening effect.

F.2 Proposition 1 under sequential timing

We extend Proposition 1 by proving a series of Lemmas. Consider the M-merger subgame.

We begin by showing that any distribution of the upstream demand between the vertically integrated firms offering the lowest input price, $w = \min_{1 \leq i \leq M} w_i$, can be sustained in equilibrium. As a first step, consider an auxiliary game, in which vertically integrated firms can price discriminate between downstream firms and downstream firms do not have access to a randomization device. Let w_k^i denote the upstream price offered by $U_i - D_i$ to D_k. We have:

Lemma F. *Consider D_k's optimal choice of upstream supplier in stage 2.5 of the above auxiliary game:*

(i) If $w_k^i = w_k^j$, then D_k is indifferent between $U_i - D_i$'s and $U_j - D_j$'s offers.

(ii) If $w_k^i > w_k \equiv \min_{1\leq j\leq M} w_k^j$, then choosing supplier $U_i - D_i$ is strictly dominated by choosing any supplier offering w_k.

Proof. Let s_l denote D_l's supplier choice. Suppose D_k purchases from $U_i - D_i$, i.e., $s_k = i$. For a given profile of supplier choices and input prices, the equilibrium downstream prices solve the following system of first-order conditions:

$$\begin{array}{llrcl} U_i - D_i & : & \frac{\gamma}{N}\left(w_k^i + \sum_{l:l\neq k, s_l=i} w_l^i\right) + 1 + \gamma\bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N})p_i, \\ U_j - D_j,\ j\neq i & : & \frac{\gamma}{N}\sum_{l:s_l=j} w_l^j + 1 + \gamma\bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N})p_j, \\ D_k & : & (w_k^i + \delta)(1 + \gamma(1 - \frac{1}{N})) + 1 + \gamma\bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N})p_k, \\ D_l,\ l\neq k & : & (w_l^{s_l} + \delta)(1 + \gamma(1 - \frac{1}{N})) + 1 + \gamma\bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N})p_l. \end{array}$$

Letting $W = \sum_{l\neq k} w_l^{s_l}$ and adding up these equations yield the equilibrium average downstream price:

$$N\left(2 + \gamma(1 - \frac{1}{N})\right)\bar{p} = N + \delta(N - M)\left(1 + \gamma(1 - \frac{1}{N})\right) + (1+\gamma)W + (1+\gamma)w_k^i.$$

It follows that D_k's equilibrium profit depends only on w_k^i and W. Hence, if $w_k^j = w_k^i$, then D_k is indifferent between $U_i - D_i$ to $U_j - D_j$. This establishes part (i).

Computing D_k's equilibrium price, output, and profit, we show that D_k's profit is strictly decreasing in w_k^i whenever D_k's demand is positive (Step 004). Therefore, D_k's profit when $w_k^i > w$ is strictly smaller than when $w_k^i = w$. By part (i), it follows that D_k is strictly better off purchasing from $U_j - D_j$ such that $w_k^j = w$, which establishes part (ii). □

As a corollary, we obtain:

Lemma G. *A profile of supplier choices is an equilibrium of stage 2.5 if and only downstream firms randomize between the suppliers offering $w \equiv \min_{1\leq i\leq M} w_i$.*

Proof. Given Lemma F–(i), the result is obvious if $w_i = w$ for all i.

Fix w and assume $w_i > w$ for some i. Fix a profile of (potentially randomized) supplier choices and suppose that, in this profile, D_k chooses $U_i - D_i$ for some realization $\tilde{\theta}$ of the sunspot. Given $\theta = \tilde{\theta}$, the supplier choices of D_k's rivals are deterministic and known to D_k. We can therefore apply Lemma F–(ii), which implies that D_k would be strictly better off choosing any firm offering w. This profile is therefore not an equilibrium.

Conversely, it is straightforward to apply Lemma F to establish that any profile of supplier choices in which downstream firms randomize between the firms offering w is indeed an equilibrium. □

Lemma G immediately implies that the existence condition for monopoly-like equilibria remains the same under sequential timing:

Lemma H. *Monopoly-like equilibria exist if and only if $\delta \geq \delta_m$.*

Proof. Immediate. □

Next, we extend the concept of collusive-like equilibria to our alternative timing with public randomization. We define the set of outcomes of the supplier choice stage as follows:

$$S = \left\{ \boldsymbol{s} \in \{0, 1, \ldots, N - M\}^M \mid \sum_{i=1}^{M} s_i = N - M \right\}.$$

In words, if the outcome of the supplier choice stage is $\mathbf{s}$, then $U_i - D_i$ supplies input to s_i unintegrated downstream firms. By Lemma F–G, an outcome S can arise in a subgame-perfect equilibrium of stage 2.5 if and only if for every i, $s_i = 0$ whenever $w_i > w = \min_{1 \leq j \leq M} w_j$. (Moreover, the analysis in the Appendix to our paper implies that we do not need to keep track of which of the unintegrated downstream firms purchase from $U_i - D_i$ $(1 \leq i \leq M)$ since $\mathbf{s}$ pins down the equilibrium profits of all firms.)

Let Σ be the symmetric group over the set $\{1, \ldots, M\}$. Σ is the set of permutations of $\{1, \ldots, M\}$. For all $(\boldsymbol{s}, \sigma) \in S \times \Sigma$, let $\boldsymbol{s}^\sigma$ denote the element of S such that $s_i^\sigma = s_{\sigma(i)}$ for all $1 \leq i \leq M$. Let $\boldsymbol{s}$, $\boldsymbol{s}' \in S$. We say that $\boldsymbol{s}$ and $\boldsymbol{s}'$ are equivalent ($\boldsymbol{s} \sim \boldsymbol{s}'$) if there exists a permutation $\sigma \in \Sigma$ such that $\boldsymbol{s}' = \boldsymbol{s}^\sigma$. Clearly, $\sim$ is an equivalence relation. Therefore, $\bar{S} = S/\sim$, the set of equivalence classes of S by $\sim$, is well defined. For all $\boldsymbol{s} \in S$, there exists a unique $[\boldsymbol{s}] \in \bar{S}$ such that $\boldsymbol{s} \in [\boldsymbol{s}]$. Let $T \subseteq S$ such that $\bigcup_{\boldsymbol{t} \in T} \{[\boldsymbol{t}]\} = \bar{S}$, and $[\boldsymbol{t}] \neq [\boldsymbol{t}']$ for all $\boldsymbol{t} \neq \boldsymbol{t}'$ in T. Then, $S = \bigcup_{\boldsymbol{t} \in T} \bigcup_{\sigma \in \Sigma} \{\boldsymbol{t}^\sigma\}$.

Let $\Delta(T \times \Sigma)$ be the set of probability measures on $T \times \Sigma$. Using Lemma G, we can redefine a collusive-like outcome at price $w > 0$ as a situation in which at least two vertically integrated firms set w, other integrated firms set prices no smaller than w, and unintegrated downstream firms randomize between the vertically integrated firms setting w. Formally, the randomization $\mu \in \Delta(T \times \Sigma)$ used by THE downstream firms must satisfy

$$\mu\left(\left\{(\boldsymbol{t}, \sigma) \in T \times \Sigma \mid \exists i \in \{1, \ldots, M\} : w_{\sigma(i)} > w \text{ and } t_{\sigma(i)} > 0\right\}\right) = 0.$$

Given such a randomization, the expected payoff of vertically integrated firm $U_i - D_i$ is:[4]

$$\mathbb{E}_\mu\left(\Pi\left(w, \frac{t_{\sigma(i)}}{N-M}\right)\right) = \sum_{\boldsymbol{t} \in T} \sum_{\sigma \in \Sigma} \mu(\boldsymbol{t}, \sigma) \Pi\left(w, \frac{t_{\sigma(i)}}{N-M}\right), \tag{2}$$

where $\Pi(\cdot, \cdot)$ is the equilibrium profit function defined in Section III(i) of the paper.

The collusive-like outcome at price w with randomization μ is an equilibrium if and only if vertically integrated firms want neither to undercut nor to exit:

$$\min_{1 \leq i \leq M} \mathbb{E}_\mu\left(\Pi\left(w, \frac{t_{\sigma(i)}}{N-M}\right)\right) \geq \max\left(\Pi(w, 0), \max_{\tilde{w} \leq w} \Pi(\tilde{w}, 1)\right).$$

We can now define the best symmetric collusive-like outcome at price w. Put

$$\alpha_s \equiv \frac{1}{N-M}\left\lfloor\frac{N-M}{M}\right\rfloor \quad \text{and} \quad \phi = 1 - (N-M)\left(\frac{1}{M} - \alpha_s\right),$$

where $\lfloor x \rfloor$ is the largest integer not greater than x. The best symmetric collusive-like outcome at price w is such that each integrated firm supplies $\alpha_s(N-M)$ firms with probability ϕ, and $\alpha_s(N-M)+1$ firms with probability $1-\phi$. Given this outcome, each vertically integrated firm earns an expected profit of

$$\Pi^b(w) = \left(1 - (N-M)\left(\tfrac{1}{M} - \alpha_s\right)\right) \Pi\left(w, \alpha_s\right) + (N-M)\left(\tfrac{1}{M} - \alpha_s\right) \Pi\left(w, \alpha_s + \tfrac{1}{N-M}\right).$$

The best symmetric collusive-like outcome at price w is an equilibrium if and only if:

$$\Pi^b(w) \geq \max\left(\Pi(w, 0), \max_{\tilde{w} \leq w} \Pi(\tilde{w}, 1)\right).$$

We prove the following lemma:

[4]Equivalently, we could have described a public randomization as a probability measure $\lambda \in \Delta S$. Using probability measures on $T \times \Sigma$ makes it easier to use symmetry arguments to prove Lemma I below.

Lemma I. *If there exists a collusive-like equilibrium at price w, then, the best symmetric collusive-like outcome at price w is also an equilibrium.*

Proof. For all $\mu \in \Delta(T \times \Sigma)$, let

$$F(\mu) = \min_{1 \le i \le M} \mathbb{E}_\mu \left(\Pi \left(w, \frac{t_{\sigma(i)}}{N-M} \right) \right).$$

Our goal is to show that the best symmetric collusive-like outcome solves the maximization problem

$$\max_{\mu \in \Delta(T \times \Sigma)} F(\mu). \tag{3}$$

Once this is shown, the lemma will follow immediately. To see this, assume the best symmetric collusive-like outcome does maximize F, and suppose there exists a collusive-like outcome with some randomization μ. Then,

$$\Pi^b(w) \ge F(\mu) = \min_{1 \le i \le M} \mathbb{E}_\mu \left(\Pi(w, \frac{t_{\sigma(i)}}{N-M}) \right) \ge \max \left(\Pi(w,0), \max_{\tilde{w} \le w} \Pi(\tilde{w},1) \right),$$

and so the best symmetric collusive-like outcome is an equilibrium.

For all $1 \le i \le M$, $\mathbb{E}_\mu \Pi \left(w, t_{\sigma(i)}/(N-M) \right)$ is linear in μ (see equation (2)). Hence, F is continuous and quasi-concave in μ. Since $\Delta(T \times \Sigma)$ is compact, it follows that maximization problem (3) has a solution.

Next, we claim that this maximization problem has a symmetric solution, i.e., there exists $\tilde{\mu} \in \Delta(T \times \Sigma)$ such that $\tilde{\mu}(\boldsymbol{t}, \sigma) = \tilde{\mu}(\boldsymbol{t}, \sigma')$ for all $\boldsymbol{t} \in T$ and $\sigma, \sigma' \in \Sigma$, and $F(\tilde{\mu}) = \max_{\mu \in \Delta(T \times \Sigma)} F(\mu)$. Let $\Delta_{sym}(T \times \Sigma)$ denote the set of symmetric probability measures. Let $\mu \in \arg\max_{\mu \in \Delta(T \times \Sigma)} F(\mu)$. For all $\hat{\sigma} \in \Sigma$, define $\mu_{\hat{\sigma}} \in \Delta(T \times \Sigma)$ as

$$\mu_{\hat{\sigma}}(\boldsymbol{t}, \sigma) = \mu(\boldsymbol{t}, \sigma \circ \hat{\sigma}) \;\; \forall \; (\boldsymbol{t}, \sigma) \in T \times \Sigma,$$

where $\circ$ is the composition operator. Then,

$$\begin{aligned}
F(\mu_{\hat{\sigma}}) &= \min_{1 \le i \le M} \left(\sum_{\boldsymbol{t} \in T} \sum_{\sigma \in \Sigma} \mu(\boldsymbol{t}, \sigma \circ \hat{\sigma}) \Pi(w, \frac{t_{\sigma(i)}}{N-M}) \right), \\
&= \min_{1 \le i \le M} \left(\sum_{\boldsymbol{t} \in T} \sum_{\sigma' \in \Sigma} \mu(\boldsymbol{t}, \sigma') \Pi(w, \frac{t_{\sigma' \circ \hat{\sigma}^{-1}(i)}}{N-M}) \right), \\
&= \min_{1 \le \hat{\sigma}(j) \le M} \left(\sum_{\boldsymbol{t} \in T} \sum_{\sigma' \in \Sigma} \mu(\boldsymbol{t}, \sigma') \Pi(w, \frac{t_{\sigma'(j)}}{N-M}) \right), \\
&= \min_{1 \le j \le M} \left(\sum_{\boldsymbol{t} \in T} \sum_{\sigma' \in \Sigma} \mu(\boldsymbol{t}, \sigma') \Pi(w, \frac{t_{\sigma'(j)}}{N-M}) \right), \\
&= F(\mu),
\end{aligned}$$

where the second and third lines follow by the changes of variables $\sigma' = \sigma \circ \hat{\sigma}$ and $j = \hat{\sigma}^{-1}(i)$, respectively.

Next, define $\tilde{\mu} = \frac{1}{|\Sigma|} \sum_{\hat{\sigma} \in \Sigma} \mu_{\hat{\sigma}}$. Clearly, $\tilde{\mu} \in \Delta_{sym}(T \times \Sigma)$. Moreover, we have that $F(\tilde{\mu}) \geq F(\mu)$ since:

$$\begin{aligned} F(\tilde{\mu}) &= F\left(\frac{1}{|\Sigma|} \sum_{\hat{\sigma} \in \Sigma} \mu_{\hat{\sigma}}\right), \\ &\geq \min_{\hat{\sigma} \in \Sigma} F(\mu_{\hat{\sigma}}) \text{ by quasi-concavity,} \\ &= F(\mu) \text{ since } F(\mu_{\hat{\sigma}}) = F(\mu) \ \forall \hat{\sigma}. \end{aligned}$$

We have thus found a symmetric probability measure that solves maximization problem (3).

Fix some $i \in \{1, \ldots, M\}$. For a given symmetric probability measure μ, we let $\mu(\boldsymbol{t}) \equiv \mu(\boldsymbol{t}, \sigma)$ for all $(\boldsymbol{t}, \sigma) \in T \times \Sigma$. Since μ is symmetric, $\mu(\boldsymbol{t})$ is well defined. Confining attention to symmetric probability measures, maximization problem (3) can then be rewritten as

$$\max_{\mu \in \Delta_{sym}(T \times \Sigma)} \left(\sum_{\boldsymbol{t} \in T} \mu(\boldsymbol{t}) \sum_{\sigma \in \Sigma} \Pi(w, \frac{t_{\sigma(i)}}{N - M}) \right), \tag{4}$$

where i is an arbitrary element of $\{1, \ldots, M\}$.

The next step is to show that the best symmetric collusive-like outcome solves maximization problem (4). Let

$$\boldsymbol{t}^b = \left(\underbrace{\alpha_s(N-M), \ldots, \alpha_s(N-M)}_{M-(N-M)(1-\alpha_s M) \text{ times}}, \alpha_s(N-M)+1, \ldots, \alpha_s(N-M)+1 \right)$$

and assume $\boldsymbol{t}^b \in T$ (if $\boldsymbol{t}^b \notin T$, then we can find the unique $\boldsymbol{t} \in T \cap [\boldsymbol{t}^b]$ and replace it by $\boldsymbol{t}^b$). The best symmetric collusive-like outcome can be generated by the following symmetric probability measure:

$$\mu^b(\boldsymbol{t}) = \begin{cases} 1 & \text{if } \boldsymbol{t} = \boldsymbol{t}^b, \\ 0 & \text{otherwise.} \end{cases}$$

We want to show that μ^b solves maximization problem (4).

Let μ be a symmetric probability measure. Define for every $1 \leq i \leq M$ and $0 \leq s \leq N-M$ $\phi_i(s) \equiv \mathbb{P}_\mu(s_i = s)$. By symmetry, $\phi_i(\cdot) = \phi(\cdot)$ for all $1 \leq i \leq M$. The expected profit of a vertically integrated firm can be rewritten as:

$$\sum_{k=0}^{N-M} \phi(k)\Pi\left(w, \frac{k}{N-M}\right).$$

We claim that if $\phi(.)$ is derived from a symmetric probability measure μ, then it must satisfy the following constraints:

(i) $\sum_{k=0}^{N-M} \phi(k) = 1$.

(ii) $\sum_{k=0}^{N-M} k\phi(k) = \frac{N-M}{M}$.

(i) is immediate. To see why (ii) has to hold, note that $\sum_{k=0}^{N-M} k\phi(k)$ is the expected number of downstream firms supplied by integrated firm $U_i - D_i$, which is equal to:

$$\sum_{\boldsymbol{s}\in S} s_i \mathbb{P}_\mu(\boldsymbol{s}) = \frac{1}{M}\sum_{j=1}^{M}\sum_{\boldsymbol{s}\in S} s_j \mathbb{P}_\mu(\boldsymbol{s}) = \frac{1}{M}\sum_{\boldsymbol{s}\in S}\left(\sum_{j=1}^{M} s_j\right)\mathbb{P}_\mu(\boldsymbol{s}) = \frac{N-M}{M}.$$

It follows that

$$\max_{(\phi(k))_{0\leq k\leq N-M}\in[0,1]^{N-M+1}} \sum_{k=0}^{N-M} \phi(k)\Pi\left(w, \frac{k}{N-M}\right) \text{ s.t. (i) and (ii) hold} \tag{5}$$

is no smaller than

$$\max_{\mu\in\Delta_{sym}(T\times\Sigma)}\left(\sum_{\boldsymbol{t}\in T}\mu(\boldsymbol{t})\sum_{\sigma\in\Sigma}\Pi\left(w, \frac{t_{\sigma(i)}}{N-M}\right)\right).$$

Next, we claim that the $\phi(\cdot)$ induced by the best symmetric collusive-like outcome solves maximization problem (5). From this, it will follow immediately that the best symmetric collusive-like outcome also solves maximization problem (4).

Define

$$h : j \in [0, N-M] \mapsto \left(\Pi\left(w, \frac{\lfloor j\rfloor + 1}{N-M}\right) - \Pi\left(w, \frac{\lfloor j\rfloor}{N-M}\right)\right)(j - \lfloor j\rfloor) + \Pi\left(w, \frac{\lfloor j\rfloor}{N-M}\right).$$

Since $\Pi(w,\cdot)$ is concave, h is concave. Moreover, we have that $h(\frac{N-M}{M}) = \Pi^b(w)$ and $h(k) = \Pi(w, \frac{k}{N-M})$ for all $k \in \{0, 1, \ldots, N-M\}$. Since h is concave, we have that for all $(\phi(k))_{0\leq k\leq N-M} \in [0,1]^{N-M+1}$ such that $\sum_{k=0}^{N-M}\phi(k) = 1$ and $\sum_{k=0}^{N-M} k\phi(k) = \frac{N-M}{M}$,

$$\Pi^b(w) = h\left(\sum_{k=0}^{N-M}\phi(k)k\right) \geq \sum_{k=0}^{N-M}\phi(k)h(k) = \sum_{k=0}^{N-M}\phi(k)\Pi\left(w, \frac{k}{N-M}\right).$$

It follows that the $\phi(\cdot)$ induced by the best symmetric collusive-like outcome solves maximization problem (5), which concludes the proof. $\square$

Next, we derive a necessary and sufficient condition for the existence of best symmetric collusive-like equilibria:

Lemma J. *There exists a threshold $\underline{\delta}_c^t \in [\underline{\delta}_c, \delta_m]$ such that there exists a best symmetric collusive-like equilibrium if and only if $\delta \in [\underline{\delta}_c^t, \overline{\delta}_c)$. When this condition is satisfied, the set of prices that can be sustained in a best symmetric collusive-like equilibrium is an interval.*

Proof. For all $\alpha \in [0, 1/M]$ and $w > 0$, define[5]

$$\tilde{\Pi}(w, \alpha) = \left(1 - (N - M)\left(\tfrac{1}{M} - \alpha\right)\right) \Pi(w, \alpha) + (N - M)\left(\tfrac{1}{M} - \alpha\right) \Pi\left(w, \alpha + \tfrac{1}{N-M}\right)$$

and

$$W(\alpha, \delta) = \left\{ w > 0 \mid \tilde{\Pi}(w, \alpha) \geq \max\left(\Pi(w, 0), \max_{\tilde{w} \leq w} \Pi(\tilde{w}, 1)\right)\right\}.$$

We will prove that for all $\alpha \in [0, 1/M]$, there exists $\underline{\delta}(\alpha)$ such that $W(\alpha, \delta)$ is a non-empty interval if and only if $\delta \in [\underline{\delta}(\alpha), \overline{\delta}_c)$. Since $\alpha_s \in [0, 1/M]$, a direct consequence of this result is that $W(\alpha_s, \lambda)$, the set of prices that can be sustained in a best symmetric collusive-like equilibrium, is a non-empty interval if and only if $\delta \in [\underline{\delta}_c^t, \overline{\delta}_c)$, where $\underline{\delta}_c^t = \underline{\delta}(\alpha_s)$.

Let $\alpha \in [0, 1/M]$. We calculate equilibrium prices, output levels and profits, as well as $\tilde{\Pi}(w, \alpha)$ (Step 005). Next, we recalculate δ_{sup} (the δ above which downstream firms cannot be active when $w = 0$), w_{max} (the w above which downstream firms cannot be active), and w_m (Step 006).

The function $\tilde{\Pi}(w, \alpha) - \Pi(w, 1)$ is strictly convex in w. It has two roots: $w = 0$ and $w = w_1$ (Step 007). Therefore, $\tilde{\Pi}(w, \alpha) > \Pi(w, 1)$ if and only if $w > \max(0, w_1)$. w_1 is strictly decreasing in δ, and there exists a threshold δ_1 such that $w_1 > 0$ if and only if $\delta < \delta_1$. Note that δ_1 does not depend on α and that it is equal to $\overline{\delta}_c$ (Step 008).

The function $\tilde{\Pi}(w, \alpha) - \Pi(w, 0)$ is strictly concave in w. It has two roots: $w = 0$ and $w = w_2$ (Step 009). It follows that, if $w_2 \leq 0$, then $\tilde{\Pi}(w, \alpha) < \Pi(w, 0)$ for all $w > 0$. Otherwise, $\tilde{\Pi}(w, \alpha) > \Pi(w, 0)$ if and only if $w \in (0, w_2)$. $w_{max} - w_2$ is decreasing in δ.

[5]Note that $\alpha + \frac{1}{N-M}$ may be strictly greater than 1. In this case, $\Pi\left(w, \alpha + \frac{1}{N-M}\right)$ no longer has an economic interpretation, but can still be computed and studied. Of course, when $\alpha = \alpha_s$, it is always the case that $\alpha + \frac{1}{N-M} \leq 1$ by construction.

When $\delta = \delta_1$, $w_2 = 0$, while $w_{max} > 0$. Therefore, $w_2 < w_{max}$ for all $\delta < \delta_1$ (step 010). $w_2 - w_1$ is non-increasing in δ and equal to zero when $\delta = \delta_1$ (Step 011). It follows that $0 < w_1 \leq w_2 < w_{max}$ when $\delta < \delta_1$, and $w_2 \leq w_1 < 0$ when $\delta > \delta_1$. Therefore, when $\delta \geq \delta_1$, $w_2 \leq 0$, so that $\tilde{\Pi}(w, \alpha) < \Pi(w, 0)$ for all $w > 0$ and $W(\alpha, \delta) = \emptyset$. Conversely, when $\delta < \delta_1$, which we assume in the following, we have $W(\alpha, \delta) \subseteq [w_1, w_2]$.

$w_m - w_1$ is strictly increasing in δ (Step 012) and positive when $\delta = \delta_1$ (Step 013). Therefore, there exists $\delta_2 < \delta_1$ such that $w_m > w_1$ if and only if $\delta > \delta_2$. $\tilde{\Pi}$ is strictly concave in w for all α, and we define w_c as the unique w such that $\partial\tilde{\Pi}/\partial w = 0$ (Step 014). $w_c - w_2$ is increasing in δ, and there exists a unique threshold δ_3 such that $w_2 < w_c$ if and only if $\delta > \delta_3$ (Step 015). Moreover, $\delta_3 < \delta_1$ (Step 016). $w_c - w_1$ has a strictly negative denominator (Step 017). Its numerator is linear and strictly decreasing in δ. Thus, $w_c - w_1$ is strictly increasing in δ and there exists a unique threshold δ_4 such that $w_c > w_1$ if and only if $\delta > \delta_4$ (Step 018).

$\delta_2 - \delta_4$ has a strictly positive denominator (Step 019) and a strictly positive numerator (Step 020): $\delta_2 > \delta_4$. Moreover, by definition of the thresholds δ_3 and δ_4 and since $w_1 < w_2$ for all $\delta < \delta_1$, $\delta_3 > \delta_4$ holds trivially.

Using the facts established above, we can characterize the set $W(\alpha, \delta)$. Assume first that $\delta_2 > \delta_3$. We distinguish several cases:

1. Assume $\delta \geq \delta_1$. Then, we already know that $W(\alpha, \lambda) = \emptyset$.

2. Next, assume $\delta \in [\delta_2, \delta_1)$. Then, $0 < w_1 \leq \min(w_m, w_2)$ and $w_2 < w_c$.

   Assume first that $w_1 < w_2 \leq w_m$. Then, for all $w \in [w_1, w_2]$,

   $$\tilde{\Pi}(w, \alpha) \geq \Pi(w, 1) = \max_{\tilde{w} \leq w} \Pi(\tilde{w}, 1),$$

   and $\tilde{\Pi}(w, \alpha) \geq \Pi(w, 0)$. Therefore, $W(\alpha, \delta) = [w_1, w_2]$.

   Conversely, if $w_2 > w_m$, then, by the same token, $[w_1, w_m] \subseteq W(\alpha, \delta)$. Besides, for all $w \in [w_m, w_2]$,

   $$\tilde{\Pi}(w, \alpha) \geq \tilde{\Pi}(w_m, \alpha) \geq \Pi(w_m, 1).$$

   Therefore, $[w_m, w_2] \subseteq W(\alpha, \delta)$, and $W(\alpha, \delta) = [w_1, w_2]$.

3. Assume $\delta \in [\delta_3, \delta_2)$. Then, $0 < w_m < w_1 < w_2 \leq w_c$, and we claim that $W(\alpha, \delta)$ is non-empty if and only if $\tilde{\Pi}(w_2, \alpha) \geq \Pi(w_m, 1)$. Indeed, if this inequality holds, then, clearly, $w_2 \in W(\alpha, \delta)$, and it is straightforward to show that $W(\alpha, \delta) =$

$[\tilde{w}_1, w_2]$, where $\tilde{w}_1$ is the unique solution of equation $\tilde{\Pi}(w,\alpha) = \Pi(w_m, 1)$ on the interval $[w_1, w_2]$. Conversely, suppose it does not hold. Since $w_1 < w_2 \leq w_c$, and since $\tilde{\Pi}(w,\alpha)$ is concave in w, $\tilde{\Pi}(w,\alpha)$ is increasing on the interval $[w_1, w_2]$. Therefore, for all $w \in [w_1, w_2]$,

$$\tilde{\Pi}(w,\alpha) \leq \tilde{\Pi}(w_2,\alpha) < \Pi(w_m, 1) = \max_{\tilde{w} \leq w} \Pi(\tilde{w}, 1),$$

and so $w \notin W(\alpha, \delta)$. It follows that $W(\alpha, \delta)$ is empty.

Define the following function:

$$d\pi_2 : \delta \in \mathbb{R} \mapsto \tilde{\Pi}(w_2, \alpha) - \Pi(w_m, 1).$$

$d\pi_2(\delta)$ is quadratic in δ (step 021). When $\delta = \delta_2$, $w_m = w_1 < w_2 < w_c$, and so $d\pi_2(\delta_2) > 0$. Conversely, when $\delta = \delta_4$, $w_m < w_c = w_1 < w_2$. Therefore,

$$\tilde{\Pi}(w_2,\alpha) < \tilde{\Pi}(w_1,\alpha) = \Pi(w_1, 1) < \Pi(w_m, 1),$$

and $d\pi_2(\delta_4) < 0$. Since $d\pi_2$ is quadratic in δ, there exists a unique threshold $R_2 \in (\delta_4, \delta_2)$, such that for all $\delta \in [\delta_4, \delta_3]$, $d\pi_2(\delta) > 0$ if $\delta > R_2$, and $d\pi_2(\delta) < 0$ if $\delta < R_2$. We can conclude that if $R_2 \leq \delta_3$, then $W(\alpha, \delta) \neq \emptyset$ for all $\delta \in [\delta_3, \delta_2)$, whereas if $R_2 > \delta_3$, then for all $\delta \in [\delta_3, \delta_2)$, $W(\alpha, \delta) \neq \emptyset$ if and only if $\delta \geq R_2$.

4. Assume $\delta \in (\delta_4, \delta_3)$. Then, $0 < w_m < w_1 < w_2 < w_c$. Following the same argument as in item 3, $W(\alpha, \delta) \neq \emptyset$ if and only if $\tilde{\Pi}(w_c, \alpha) \geq \Pi(w_m, 1)$. Moreover, when this condition holds, $W(\alpha, \delta) = [\tilde{w}_1, \tilde{w}_2]$, where $\tilde{w}_1$ and $\tilde{w}_2$ are the unique solutions of equation $\tilde{\Pi}(w,\alpha) = \Pi(w_m, 1)$ on the intervals $[w_1, w_c]$ and $[w_c, w_2]$, respectively.

   Define the following function:

   $$d\pi_c : \delta \in \mathbb{R} \mapsto \tilde{\Pi}(w_c, \alpha) - \Pi(w_m, 1).$$

   $d\pi_c$ is quadratic in δ (step 022) and it is straightforward to show that $d\pi_c(\delta_4) < 0$.

   Assume first that $R_2 < \delta_3$. Then, we know that $d\pi_2(\delta_3) > 0$. Since, when $\delta = \delta_3$, $w_2 = w_c$, it follows that $d\pi_c(\delta_3) = d\pi_2(\delta_3) > 0$. Since $d\pi_c$ is quadratic, $d\pi_c(\delta_4) < 0$ and $d\pi_c(\delta_3) > 0$, it follows that there exists a unique threshold $R_c \in (\delta_4, \delta_3)$ such that for all $\delta \in [\delta_4, \delta_3]$, $d\pi_c(\delta) > 0$ if $\delta > R_c$, and $d\pi_c(\delta) < 0$ if $\delta < R_c$. Therefore, for all $\delta \in (\delta_4, \delta_3)$, $W(\alpha, \delta) \neq \emptyset$ if and only if $\delta \geq R_c$.

   Conversely, assume $R_2 \geq \delta_3$. Then, $d\pi_c(\delta_3) \leq 0$. Besides, when $\delta = \delta_2 (> \delta_3)$, $d\pi_c(\delta) > d\pi_2(\delta) > 0$. Since $d\pi_c$ is quadratic in δ and strictly negative when $\delta = \delta_4$, it follows that $d\pi_c(\delta) < 0$ for all $\delta \in (\delta_4, \delta_3)$. Therefore, $W(\alpha, \delta)$ is empty for all $\delta \in (\delta_4, \delta_3)$.

5. Assume $\delta \leq \delta_4$. Then, $w_m \leq w_c < w_1 < w_2$ or $w_m < w_c \leq w_1 < w_2$. In both cases, $\Pi(w_m, 1) > \tilde{\Pi}(w, \alpha)$ for all $w \in [w_1, w_2]$, and so $W(\alpha, \delta)$ is empty.

We can conclude that, if $R_2 \geq \delta_3$, then $W(\alpha, \delta) \neq \emptyset$ if and only if $\delta \in [R_2, \delta_1)$. If $R_2 < \delta_3$, then $W(\alpha, \lambda) \neq \emptyset$ if and only if $\delta \in [R_c, \delta_1)$. Moreover, when $W(\alpha, \delta)$ is not empty, it is an interval.

Conversely, if $\delta_2 \leq \delta_3$, then it is straightforward to adapt the above reasoning to show that there exists a threshold R_c such that $W(\alpha, \delta) \neq \emptyset$ if and only if $\delta \in [R_c, \delta_1)$. This concludes the proof of the existence of the thresholds $\underline{\delta}_c^t$ and $\overline{\delta}_c$.

Finally, we show that $\underline{\delta}_c^t \in [\underline{\delta}_c, \delta_m]$. Suppose $\delta = \delta_m$. Then, $\Pi(w_m, 1) = \Pi(w_m, 0)$. By concavity, it follows that

$$\Pi\left(w_m, \alpha_s + \frac{1}{N-M}\right) \geq \Pi(w_m, 1) = \Pi(w_m, 0),$$

and that

$$\Pi(w_m, \alpha_s) \geq \Pi(w_m, 1) = \Pi(w_m, 0).$$

This implies in particular that

$$\phi\pi(w_m, \alpha_s) + (1-\phi)\Pi\left(w_m, \alpha_s + \frac{1}{N-M}\right) \geq \max\left(\Pi(w_m, 0), \Pi(w_m, 1)\right).$$

Therefore, there exists a best symmetric collusive-like equilibrium at price w_m, and $\underline{\delta}_c^t \leq \delta_m$.

The inequality $\underline{\delta}_c^t \geq \underline{\delta}_c$ follows from the fact that, since $\Pi(w, \alpha)$ is concave in α,

$$\Pi\left(\frac{1}{M}, w\right) \geq \phi\pi(w, \alpha_s) + (1-\phi)\Pi\left(w, \alpha_s + \frac{1}{N-M}\right). \qquad \square$$

Combining the above lemmas, we obtain the following proposition:

Proposition D. *Under sequential timing, in the M-merger subgame, there exists a threshold $\underline{\delta}_c^t(\gamma) \in [\underline{\delta}_c(\gamma), \delta_m(\gamma)]$ such that:*

(i) There is a monopoly-like equilibrium if and only if $\delta \geq \delta_m(\gamma)$.

(ii) There is a best symmetric collusive-like equilibrium if and only if $\underline{\delta}_c^t(\gamma) \leq \delta < \overline{\delta}_c(\gamma)$. In this case, the set of prices that can be sustained in a best symmetric collusive-like equilibrium is an interval.

(iii) If $\delta < \underline{\delta}_c(\gamma)$, then the Bertrand outcome is the only equilibrium.

G Proof of Proposition 4

We consider the M-merger subgame. All the calculations for this section are in the Mathematica notebook `05_discrimination.nb`. As in the paper, $\Pi(w, \alpha)$ denotes the profit of a vertically integrated firm with upstream market share α when the input price is w. For $1 \leq k \leq N-M$, and $w_{M+1}, \dots, w_{M+k}, \hat{w} \in \mathbb{R}$, we denote by $\Pi((w_{M+1}, \dots, w_{M+k}), \hat{w}, k)$ the equilibrium profit of a vertically integrated firm when it supplies firms $D_{M+1}, \dots, D_{M+k}$ at prices $w_{M+1}, \dots, w_{M+k}$ and the rest of the upstream market is supplied by other vertically integrated firms at price $\hat{w}$.[6] Let $\bar{w} > 0$. We consider the following maximization problem:

$$\begin{array}{c} \max_{(w_{M+1},\dots,w_{M+k})} \Pi((w_{M+1}, \dots, w_{M+k}), \hat{w}, k) \\ \text{s.t. } w_{M+j} \leq \bar{w} \text{ for all } 1 \leq j \leq k. \end{array} \tag{6}$$

Lemma K. *Maximization problem* (6) *has a unique solution, which is symmetric.*

Proof. Let $k \geq 2$. Suppose $U_i - D_i$ supplies the first k downstream firms at prices $w_{M+1}, \dots, w_{M+k}$. Assume without loss of generality that the other downstream firms purchase from $U_{i'} - D_{i'}$ at price $\hat{w}$. We claim that, if there exist $1 \leq j' < j'' \leq k$ such that $w_{M+j'} \neq w_{M+j''}$, then $U_i - D_i$ can achieve a strictly higher equilibrium profit by offering $\tilde{w} \equiv \sum_{j=1}^{k} w_{M+j}/k$ to the k downstream firms. If $U_i - D_i$ sticks to the input price vector $(w_{M+1}, \dots, w_{M+k})$, then the first order conditions in the downstream market are:

$$\begin{array}{lcrcl}
U_i - D_i & : & \frac{\gamma}{N}(\sum_{j=1}^{k} w_{M+j}) + 1 + \gamma\bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N})p_i, \\
U_{i'} - D_{i'} & : & \frac{\gamma}{N}(N - M - k)\hat{w} + 1 + \gamma\bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N})p_{i'}, \\
U_{i''} - D_{i''},\ i'' \neq i, i' & : & 1 + \gamma\bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N})p_{i''}, \\
D_{M+j},\ 1 \leq j \leq k & : & (w_{M+j} + \delta)(1 + \gamma(1 - \frac{1}{N})) + 1 + \gamma\bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N})p_{M+j}, \\
D_{M+j},\ j > k & : & (\hat{w} + \delta)(1 + \gamma(1 - \frac{1}{N})) + 1 + \gamma\bar{p} & = & (2(1+\gamma) - \frac{\gamma}{N})p_{M+j}.
\end{array}$$

Adding up these first-order conditions, we obtain the equilibrium average downstream price $\bar{P}$. It is easily shown that $\bar{P}$ depends on $(w_{M+1}, \dots, w_{M+k})$ only through $\tilde{w}$. It follows that the equilibrium average downstream price remains the same when $U_i - D_i$ deviates to the uniform input price $\tilde{w}$.

Let $(P_j)_{1 \leq j \leq N}$ (resp. $(Q_j)_{1 \leq j \leq N}$) be the equilibrium downstream price (resp. quantity) vector when $U_i - D_i$ prices asymmetrically, and $(\tilde{P}_j)_{1 \leq j \leq N}$ (resp. $(\tilde{Q}_j)_{1 \leq j \leq N}$) the

[6] It is straightforward to adapt the analysis in the Appendix to our paper to show that this profit does not depend on how the rest of the market is shared among other integrated firms. The function Π is therefore well defined.

equilibrium downstream price (resp. quantity) vector when it prices uniformly. Since $U_i - D_i$'s first-order condition and $\bar{P}$ remain the same in both scenarios, it follows that $P_i = \tilde{P}_i$ and $Q_i = \tilde{Q}_i$. Moreover, under uniform input pricing, the k downstream firms set the same downstream price. Therefore, $\tilde{Q}_{M+j} = \tilde{Q}$ for all $1 \le j \le k$. The variation in $U_i - D_i$'s equilibrium profit when it deviates to uniform pricing, $\Delta\pi$, is therefore given by:

$$
\begin{aligned}
\Delta\pi &= \sum_{j=1}^{k} \left(\tilde{w}\tilde{Q} - w_{M+j}Q_{M+j}\right), \\
&= \tilde{Q}\sum_{j=1}^{k}(\tilde{w} - w_{M+j}) + \sum_{j=1}^{k} w_{M+j}(\tilde{Q} - Q_{M+j}), \\
&= \sum_{j=1}^{k} w_{M+j}(\tilde{Q} - Q_{M+j}), \\
&= \frac{1+\gamma}{N}\sum_{j=1}^{k} w_{M+j}(P_{M+j} - \tilde{P}), \text{ using the demand function,} \\
&= \frac{(1+\gamma)(1+\gamma-\gamma/N)}{N(2(1+\gamma)-\gamma/N)}\sum_{j=1}^{k} w_{M+j}(w_{M+j} - \tilde{w}), \text{ using first-order conditions,} \\
&= \frac{(1+\gamma)(1+\gamma-\gamma/N)}{N(2(1+\gamma)-\gamma/N)}\sum_{j=1}^{k}(w_{M+j} - \tilde{w})^2 > 0.
\end{aligned}
$$

Maximization problem (6) therefore boils down to

$$\max_{w} \Pi\left(w, \hat{w}, \frac{k}{N-M}\right) \text{ s.t. } w \le \bar{w},$$

where

$$\Pi\left(w, \hat{w}, \frac{k}{N-M}\right) \equiv \Pi\left((w, \ldots, w), \hat{w}, k\right).$$

In the Mathematica notebook (Step 003), we show that $\Pi(\cdot, \hat{w}, \alpha)$ is strictly concave, which implies that this problem has a unique solution. □

Recall that w_m is the monopoly upstream price under uniform pricing. Using Lemma K with $k = N - M$ and $\bar{w} = \bar{m}$ and the definition of w_m, we can conclude that the monopoly upstream price vector is $(w_m, \ldots, w_m)$. Lemma K also implies that when a vertically integrated firm undercuts from a monopoly-like or a collusive-like equilibrium candidate, it finds it optimal to offer the same upstream price to all the unintegrated downstream firms it targets with its deviation. For $\alpha \in \{0, \frac{1}{N-M}, \ldots, \frac{N-M-1}{N-M}, 1\}$, we denote by $\Pi(w, \hat{w}, \alpha)$ the profit of a vertically integrated firm when it supplies $\alpha(N-M)$ downstream firms at price w and the rest of the upstream market is supplied by the other vertically integrated firms at price $\hat{w}$.

Lemma L. *There exists a monopoly-like equilibrium if and only if*

$$\Pi(w_m, 0) \geq \max_{\substack{\alpha \in \{\frac{1}{N-M}, \ldots, \frac{N-M-1}{N-M}, 1\} \\ w \leq w_m}} \Pi(w, w_m, \alpha). \tag{7}$$

There exists a symmetric collusive-like equilibrium at price w if and only if[7]

$$\Pi(w, \frac{1}{M}) \geq \max \left(\Pi(0, w), \max_{\substack{\alpha \in \{\frac{1}{N-M}, \ldots, \frac{N-M-1}{N-M}, 1\} \\ \tilde{w} \leq w}} \Pi(\tilde{w}, w, \alpha) \right). \tag{8}$$

Proof. Clearly, for a monopoly-like equilibrium to exist, condition (7) must hold. Conversely, suppose condition (7) holds. Assume $U_i - D_i$ supplies the upstream market at price w_m, and consider the deviation incentives of $U_j - D_j$. Following a deviation by $U_j - D_j$, we select an equilibrium of the continuation subgame in which a downstream firm that receives an offer from $U_j - D_j$ with an input price no smaller than w_m purchases from $U_i - D_i$ with probability 1. With this selection and using Lemma K, $U_j - D_j$ cannot earn more than the term on the right-hand side of equation (7), and this deviation is therefore not profitable. The second part of the lemma follows from a similar argument. □

Using Lemma L, we prove the following lemmas:

Lemma M. *For all (M, N, γ), there exist thresholds $N^d(M, \gamma) > M+3$ and $\delta_m^d(M, N, \gamma) \geq \delta^m(M, N, \gamma)$ such that monopoly-like equilibria exist if and only if $N \leq N_m^d(M, \gamma)$ and $\delta \geq \delta_m^d(M, N, \gamma)$.*

Proof. We begin by extending the domain of $\Pi(w, \hat{w}, \cdot)$ to $[0, 1]$. Let $\alpha \in \{0, \frac{1}{N-M}, \ldots, 1\}$, and suppose that $U_i - D_i$ supplies $\alpha(N-M)$ downstream firms at price w while $U_j - D_j$ supplies $(1-\alpha)(N-M)$ downstream firms at price $\hat{w}$. The equilibrium downstream prices are denoted as follows: $\bar{P}(w, \hat{w}, \alpha)$ is the average downstream price; $P_i(w, \hat{w}, \alpha)$, $P_j(w, \hat{w}, \alpha)$, $P_k(w, \hat{w}, \alpha)$, $P_{di}(w, \hat{w}, \alpha)$ and $P_{dj}(w, \hat{w}, \alpha)$ are the downstream prices charged by $U_i - D_i$, $U_j - D_j$, vertically integrated firms $U_k - D_k$ such that

[7] In a symmetric collusive-like equilibrium, all vertically integrated firms offer the same input price to all $N-M$ unintegrated downstream firms.

$k \neq i, j$, downstream firms D_{di} supplied by $U_i - D_i$, and downstream firms D_{dj} supplied by $U_j - D_j$, respectively. These prices solve the following system of equations:

$$
\begin{aligned}
0 &= 1 + \gamma\bar{P}(w,\hat{w},\alpha) + \frac{\gamma}{N}\alpha(N-M)w - (2(1+\gamma) - \frac{\gamma}{N})P_i(w,\hat{w},\alpha),\\
0 &= 1 + \gamma\bar{P}(w,\hat{w},\alpha) + \frac{\gamma}{N}(1-\alpha)\alpha(N-M)\hat{w} - (2(1+\gamma) - \frac{\gamma}{N})P_j(w,\hat{w},\alpha),\\
0 &= 1 + \gamma\bar{P}(w,\hat{w},\alpha) - (2(1+\gamma) - \frac{\gamma}{N})P_k(w,\hat{w},\alpha),\\
0 &= 1 + \gamma\bar{P}(w,\hat{w},\alpha) + (w+\delta)(1+\gamma(1-\frac{1}{N})) - (2(1+\gamma) - \frac{\gamma}{N})P_{di}(w,\hat{w},\alpha),\\
0 &= 1 + \gamma\bar{P}(w,\hat{w},\alpha) + (\hat{w}+\delta)(1+\gamma(1-\frac{1}{N})) - (2(1+\gamma) - \frac{\gamma}{N})P_{dj}(w,\hat{w},\alpha),
\end{aligned}
$$

$$
\bar{P}(w,\hat{w},\alpha) = \frac{P_i(w,\hat{w},\alpha)+P_j(w,\hat{w},\alpha)+(M-2)P_k(w,\hat{w},\alpha)+\alpha(N-M)P_{di}(w,\hat{w},\alpha)+(1-\alpha)(N-M)P_{dj}(w,\hat{w},\alpha)}{N}.
$$

This system still yields a unique solution for any $\alpha \in [0,1]$. The domain of the functions $\bar{P}(w,\hat{w},\cdot)$ and $P_l(w,\hat{w},\cdot)$ ($l = i, j, k, di, dj$) can therefore be extended to $[0,1]$. Let $Q_i(w,\hat{w},\alpha)$ denote the downstream demand of firm $U_i - D_i$ and $Q_{di}(w,\hat{w},\alpha)$ the downstream demand of a downstream firm purchasing from $U_i - D_i$, evaluated at those downstream equilibrium prices. Again, the domain of these functions can be safely extended to $\alpha \in [0,1]$. Then, the function

$$
\Pi(w,\hat{w},\alpha) = P_i(w,\hat{w},\alpha)Q_i(w,\hat{w},\alpha) + w\alpha(N-M)Q_{di}(w,\hat{w},\alpha)
$$

is also well defined and smooth on $[0,1]\times\mathbb{R}^2$. This function is defined in Step 001 of the Mathematica file.

Next, we recalculate the monopoly upstream price w_m and the threshold δ_m for the existence of monopoly-like equilibria in the uniform pricing case. δ_{sup} is defined as the threshold above which the monopoly upstream price is no longer interior (Step 002). We restrict attention to values of δ smaller than δ_{sup} in the following. If $\delta < \delta_m$, then there are clearly no monopoly-like equilibria, with or without price discrimination.

Next, suppose $\delta \in [\delta_m, \delta_{sup})$. For all $\alpha \in (0,1]$, $\Pi(w, w_m, \alpha)$ is strictly concave in w, and there exists a unique $w_{dev}(\alpha)$ that solves the first-order condition $\partial\Pi/\partial w = 0$ (Step 003). We show that $w_{dev}(\alpha) < w_m$ for all $\alpha \in (0,1)$ (Step 004). Define

$$
\Delta\Pi(\alpha) \equiv \Pi(w_{dev}(\alpha), w_m, \alpha) - \Pi(w_m, 0).
$$

Since $w_{dev}(\alpha) \leq w_m$, Condition (7) is equivalent to $\Delta\Pi(\alpha) \leq 0$ for all $\alpha \in \{\frac{1}{N-M}, \ldots, 1\}$. $\Delta\Pi$ can be written as $R(\alpha)/O(\alpha)$, where R and O are polynomials in $(\alpha, \delta, \gamma, M, N)$, and $O(\alpha) > 0$ for all α (Step 005). R can be rewritten as $R(\alpha) = \alpha\kappa(\delta)S(\alpha)$, where S is a polynomial in $(\alpha, \delta, \gamma, M, N)$, and $\kappa > 0$ (Step 006). S is strictly convex in α (Step

007). Moreover, since $\Delta\Pi(1) \leq 0$, we have that $S(1) \leq 0$. It follows that Condition (7) is equivalent to $S(\frac{1}{N-M}) \leq 0$.

In the following, we study the sign of $T \equiv S(\frac{1}{N-M})$. T is strictly decreasing in δ (Step 008). At $\delta = \delta_{sup}$, we have that $T < 0$ if and only if $f(M, N, \gamma) > 0$, where f is a polynomial (Step 009). Moreover, $f(M, M+1, \gamma) > 0$, and $\lim_{N\to\infty} f(M, N, \gamma) = -\infty$ (Step 010). Therefore, by continuity, the equation $f(M, N, \gamma) = 0$ has a solution in N on $(M+1, \infty)$.

The next step is to show that this solution is unique for given (M, γ). We do so by proving that $\partial f(M, N, \gamma)/\partial N < 0$ whenever $f(M, N, \gamma) = 0$. We show that $f(M, N, \gamma)$ is strictly concave in M and has two roots, $M = M_1$ and $M = M_2$, which satisfy $M_2 < N - 1 < M_1$ (Step 011). The relevant root for us is thus M_2. We show that $M_2 \geq 2$ if and only if N is greater or equal to some threshold $\overline{N}$ (Step 012). Condition $f(M, N, \gamma) = 0$ therefore reduces to $M = M_2$ and $N \geq \overline{N}$. When those conditions are satisfied, we find that

$$\left.\frac{\partial f(M, N, \gamma)}{\partial N}\right|_{M=M_2} < 0 \text{ (Step 013).}$$

It follows that, for given (M, γ), the equation $f(M, N, \gamma) = 0$ has a unique solution, $N_m^d(M, \gamma)$. We also show that $f(M, M+3, \gamma) > 0$ (Step 014), which implies that $N_m^d(M, \gamma) > M + 3$ for all (M, γ).

We can conclude that, if $N < N_m^d(M, \gamma)$, then there exists $\delta_m^d \in [\delta_m, \delta_{sup})$ such that monopoly-like equilibria exist if and only if $\delta \geq \delta_m^d$. Otherwise, there are no monopoly-like equilibria. □

Lemma N. *There exists a threshold $\underline{\delta}_c^d(M, N, \gamma) \in [\underline{\delta}_c(M, N, \gamma), \bar{\delta}_c(M, N, \gamma))$ such that a symmetric collusive-like equilibrium exists if $\delta \in [\underline{\delta}_c^d(M, N, \gamma), \bar{\delta}_c(M, N, \gamma))$.*

Proof. We begin by redefining the upper bound for collusive-like equilibrium in the uniform pricing case, $\bar{\delta}_c$ (Step 015). If $\delta \geq \bar{\delta}_c$, then firms always want to exit, and so there are no collusive-like equilibria. In the following, we assume $\delta < \bar{\delta}_c$. We focus on a particular symmetric collusive-like equilibrium candidate at price w_c, where w_c is such that the function $f : \alpha \in [0, 1] \mapsto \Pi(w, w, \alpha)$ is maximized at $\alpha = 1/M$ when $w = w_c$. For all $w > 0$, the function f is strictly concave in α, and there is a unique $w > 0$ such that $f'(1/M) = 0$, which we denote w_c (Step 016).

For all $\alpha > 0$, we know that $\Pi(w, w_c, \alpha)$ is concave in w. Define the unconstrained deviation price, $w_{dev}^c(\alpha) = \arg\max_w \Pi(w, w_c, \alpha)$, and let $\Delta w(\alpha) = w_{dev}^c(\alpha) - w_c$. We show that $\Delta w(0) \geq 0$ if and only if $\delta \geq \underline{\delta}_c^d$, where the threshold $\underline{\delta}_c^d$ is strictly smaller than $\bar{\delta}_c$ (Step 017). In the following, we assume $\delta \in [\underline{\delta}_c^d, \bar{\delta}_c)$. We show that $\Delta w'(\alpha) \geq 0$ for all α (Step 018). It follows that $w_{dev}^c(\alpha) \geq w_c$ for all α, which, by concavity, implies that $\max_{w \leq w_c} \Pi(w, w_c, \alpha) = \Pi(w_c, \alpha)$ for all α. It follows that condition (8) is satisfied since:

$$\begin{aligned}
\Pi\left(w_c, \frac{1}{M}\right) &= \max\left(\Pi(w_c, 0), \max_{\alpha\in[0,1]} \Pi(w_c, w_c, \alpha)\right) \text{ by definition of } w_c, \\
&= \max\left(\Pi(w_c, 0), \max_{\alpha\in[0,1]} \left\{\max_{w \leq w_c} \Pi(w, w_c, \alpha)\right\}\right), \\
&\geq \max\left(\Pi(w_c, 0), \max_{\substack{\alpha\in\{1/(N-M),\dots,1\} \\ w\leq w_c}} \Pi(w, w_c, 0)\right). \qquad \square
\end{aligned}$$

Combining Lemmas M and N, we obtain Proposition 4 .

H Two-Part Tariffs

In this section we assume that upstream firms compete in two-part tariffs, confining attention to the case where fixed parts are non-negative.

As mentioned in the paper, we assume that (observable) supplier choices are made before downstream prices are set, as in Section F. If we were to stick to the simultaneous timing used in the baseline model, we would face the following problem. Suppose U_i offers a low variable part and a high fixed part, whereas U_j offers a high w and a low T. Then, a downstream firm's optimal choice of supplier may depend on the downstream price it sets. If it sets a low downstream price, then the demand it receives is high, incentives to minimize marginal cost are strong, and so the downstream firm should pick U_i's contract. Conversely, if it sets a high price, then it should choose U_j's contract. This mechanism implies that a firm's marginal cost can be an increasing and discontinuous function of its price, which can make that firm's best-response function discontinuous and/or non-convex-valued. Such complications are likely to jeopardize equilibrium existence in stage 3.

Lemmas E and F imply that the equilibrium profit of an unintegrated downstream firm (gross of the fixed part) is a decreasing function of the variable part of the tariff it chooses. This implies that the Bertrand outcome is always an equilibrium, regardless of how many mergers have taken place.

We establish the existence condition for monopoly-like equilibria when $N = M+1$ in Section H.1. The reason why we restrict attention to the case $N = M+1$ is the following. If there are multiple unintegrated downstream firms in the merger-wave subgame, then, starting from a monopoly-like outcome, a deviating vertically integrated firm may find it optimal to offer a contract that attracts fewer than $N-M+1$ unintegrated downstream firms. This can be done by choosing a contract with a low variable part and a high fixed part, as a downstream firm is more likely to find such a contract attractive when downstream prices are high, which arises when no other downstream firm accepts the contract. It is hard to check whether such a deviation is profitable in the general case with an arbitrary number of upstream and downstream firms.

Finally, in Section H.2, we provide a sufficient condition for collusive-like equilibria to exist, focusing on the case where $M = 2$ and $N = 4$ for simplicity. Note that even in that special case, we need to account for deviations of the type described in the previous paragraph.

H.1 Proof of Proposition 5

Proof. Consider the merger-wave subgame. We start by showing that a monopoly-like equilibrium exists if and only if $\Pi(w_m^{tp}, 1) \leq \Pi(w_m^{tp}, 0)$. Suppose a monopoly-like equilibrium exists and let $T^{tp} \geq 0$ denote the fixed part of the contract accepted by the unintegrated downstream firm in equilibrium. It must be that the firms that do not supply the upstream market are not willing to undercut:

$$\Pi(w_m^{tp}, 0) \geq \Pi(w_m^{tp}, 1) + T^{tp} \geq \Pi(w_m^{tp}, 1),$$

where the second inequality follows as $T^{tp} \geq 0$.

Conversely, suppose $\Pi(w_m^{tp}, 1) \leq \Pi(w_m^{tp}, 0)$. We distinguish two cases. Assume first that $\Pi(w_m^{tp}, 1) + T_m^{tp} \leq \Pi(w_m^{tp}, 0)$. Then, the monopoly-like outcome in which $U_1 - D_1$ offers (w_m^{tp}, T_m^{tp}) and other vertically integrated firms make no offer is an equilibrium. Second, assume that $\Pi(w_m^{tp}, 1) \leq \Pi(w_m^{tp}, 0) < \Pi(w_m^{tp}, 0) + T_m^{tp}$. Then, the monopoly-like outcome in which all integrated firms offer $(w_m^{tp}, \Pi(w_m^{tp}, 0) - \Pi(w_m^{tp}, 1))$ and the unintegrated downstream firm accepts $U_1 - D_1$'s contract is an equilibrium.

Next, we find a necessary and sufficient condition for $\Pi(w_m^{tp}, 0) \geq \Pi(w_m^{tp}, 0)$. All calculations are in the Mathematica notebook `07_two_part_monopoly.nb`.

We compute equilibrium downstream prices, quantities, and profits when $U_i - D_i$ supplies D_{M+1} at price w (Step 001). The joint profit of $U_i - D_i$ and D_{M+1}, $\Pi(w, 1) + \Pi_d(w)$ is concave in w and reaches its maximum at $\hat{w}_m^{tp}$ (Step 002). We also define the thresholds δ_{sup} and w_{max} (Step 003): w_{max} is the upstream price threshold above which downstream firms cannot be active; δ_{sup} is such that $\hat{w}_m^{tp} < w_{max}$ if and only if $\delta < \delta_{sup}$. In the following, we assume $\delta < \delta_{sup}$. By choosing a high enough $\overline{m}$, we can ensure that $\Pi_d(\hat{w}_m^{tp}) \geq \overline{\pi}^d$. We therefore have $w_m^{tp} = \hat{w}_m^{tp}$.

$\Pi(w_m^{tp}, 0) - \Pi(w_m^{tp}, 1)$ is strictly concave in δ, and positive for $\delta = \delta_{sup}$. Therefore, there exists $\delta_m^{tp} < \delta_{sup}$ such that $\Pi(w_m^{tp}, 0) \geq \Pi(w_m^{tp}, 1)$ if and only if $\delta \geq \delta_m^{tp}$ (Step 004). Finally, we check that δ_m^{tp} is larger than δ_m, the threshold for monopoly-like equilibria in the linear-tariff case (Step 005). □

H.2 Proof of Proposition 6

Proof. All calculations are in the Mathematica notebook `08_two_part_collusive.nb`. We compute equilibrium downstream prices, quantities, and profits for all possible outcomes in the upstream market (Step 001). We also define three thresholds: δ_{sup}, δ_{inf} and w_{max} (Step 002). δ_{sup} (resp. δ_{inf}) is the threshold above which (resp. below which) unintegrated downstream firms (resp. vertically integrated firms) cannot be active when $w = 0$. When $\delta_{inf} < \delta < \delta_{sup}$, w_{max} is the upstream price threshold above which downstream firms cannot be active. In the following, we assume $\delta_{inf} < \delta < \delta_{sup}$.

Since the two unintegrated downstream firms do not necessarily purchase the input at the same variable part, we need to introduce additional notation:

- $\Pi(w, w', \frac{1}{2}, \frac{1}{2})$ is the profit (gross of fixed fees) of a vertically integrated firm that supplies one downstream firm with a variable part of w when the other downstream firm buys at a variable part of w' from the other integrated firm;

- $\Pi_d(w, w')$ is the profit (gross of the fixed fee) of a downstream firm that buys the input at marginal price w from a vertically integrated firm, when its downstream rival buys at w' from a vertically integrated firm (recall that the identity of vertically integrated upstream suppliers does affect the downstream firms' profits);

- $\Pi_d(\bar{m}, w', 0, \frac{1}{2})$ is the profit of a downstream firm when it buys the input from the alternative supplier and its downstream competitor buys at w' from a vertically integrated firm.

In the candidate collusive-like equilibrium, both vertically integrated firms offer (w, T), $w > m$, D_3 buys from $U_1 - D_1$, and D_4 buys from $U_2 - D_2$. This is an equilibrium if the following deviations are not profitable:

(i) A vertically integrated firm does not want to *undercut selectively*, i.e., change its upstream offer and attract only one downstream firm;

(ii) a vertically integrated firm does not want to *undercut*, i.e., change its upstream offer and attract both downstream firms;

(iii) a vertically integrated firm does not want to *exit*, i.e., withdraw its upstream offer;

(iv) a *downstream firm* does not want to deviate by accepting several offers or switching to the alternative supplier.

We derive sufficient conditions for (i)–(iv).

(i) No selective undercutting. Consider the following deviation: $U_1 - D_1$ offers (w', T') in order to attract only one downstream firm, say D_3. A necessary condition for D_3 to accept the deviation is

$$\Pi_d(w', w) - T' \geq \Pi_d(w, w) - T. \tag{9}$$

Note that this is not a sufficient condition for the deviation to be feasible since it must also be that $T' \geq 0$ and that D_4 still wants to purchase from $U_2 - D_2$. This is not a problem since we are only looking for a sufficient condition for (i). Condition (9) imposes an upper bound on T', and the deviation profit is therefore bounded above by

$$\max_{w'} \Pi(w', w, \frac{1}{2}, \frac{1}{2}) + T + \Pi_d(w', w) - \Pi_d(w, w).$$

A sufficient condition for (i) to hold is that is that the maximum is smaller than $\Pi(w, w, \frac{1}{2}, \frac{1}{2}) + T$. This condition is equivalent to

$$w \in \arg\max_{w'} \Pi(w', w, \frac{1}{2}, \frac{1}{2}) + \Pi_d(w', w).$$

For any w, $w' \mapsto \Pi(w', w, \frac{1}{2}, \frac{1}{2}) + \Pi_d(w', w)$ is concave in w' and reaches its maximum at some $w' = f(w)$. The function $w \mapsto f(w)$ has a unique fixed point, w_c, which satisfies $w_c > m$ (Step 003). A sufficient condition for (i) is therefore

$$w = w_c. \tag{10}$$

There exists δ_0 such that $w_c \leq w_{max}$ if and only if $\delta \leq \delta_0$ (Step 004). We assume $\delta \leq \delta_0$ and $w = w_c$ from now on.

(ii) No undercutting. Consider a deviation in which $U_1 - D_1$ sets (w', T') and attracts both downstream firms. A necessary condition for there to be an equilibrium in which both downstream firms accept the deviating offer is

$$\Pi_d(w', w') - T' \geq \Pi_d(w_c, w') - T.$$

This imposes an upper bound for T', so that the deviation profit is bounded above by

$$\arg\max_{w'} \Pi(w', 1) + 2[T + \Pi_d(w', w') - \Pi_d(w_c, w')].$$

The objective function is concave in w' and achieves its maximum at $w' = w_u$ (Step 005). A sufficient condition for (ii) is thus

$$\Pi(w_u, 1) + 2[T + \Pi_d(w_u, w_u) - \Pi_d(w_c, w_u)] \leq \Pi(w_c, w_c, \frac{1}{2}, \frac{1}{2}) + T,$$

which can be rewritten as (Step 006)

$$T \leq \Pi(w_c, w_c, \frac{1}{2}, \frac{1}{2}) + 2\Pi_d(w_c, w_u) - 2\Pi_d(w_u, w_u) - \Pi(w_u, 1) \equiv T_1. \tag{11}$$

(iii) No exit. A vertically integrated firm does not want to withdraw its upstream offer if and only if

$$\Pi(w_c, 0) \leq \Pi(w_c, w_c, \frac{1}{2}, \frac{1}{2}) + T,$$

which can be rewritten as (Step 007)

$$T \geq \Pi(w_c, 0) - \Pi(w_c, w_c, \frac{1}{2}, \frac{1}{2}) \equiv T_0. \tag{12}$$

(iv) Downstream firms. We first need to specify a value for $\overline{m}$. We set $\overline{m}$ such that a downstream firm receives zero profit if both downstream firms buy from the alternative source of input (Step 008). When $w = w_c$, condition (iv) holds if and only if no downstream firm wants to accept several offers,

$$T \geq 0, \tag{13}$$

or switch to the alternative supplier (Step 009):

$$T \leq \Pi_d(w_c, w_c) - \Pi_d(\bar{m}, w_c, 0, \frac{1}{2}) \equiv T_2. \tag{14}$$

Summary. Combining conditions (10)–(14), a sufficient condition for the candidate equilibrium to be an equilibrium is $w = w_c$ and $\max(0, T_0) \leq T \leq \min(T_1, T_2)$. Therefore, there exists a collusive-like equilibrium if $\max(0, T_0) \leq \min(T_1, T_2)$. We show that:

- there exists $\delta_1 < \delta_0$ such that $T_1 \geq 0$ if and only if $\delta \geq \delta_1$ (Step 010);
- there exists $\delta_2 < \delta_0$ such that $T_2 \geq 0$ if and only if $\delta \leq \delta_2$ (Step 011);

- $T_0 \leq T_1$ (Step 012);
- there exists $\delta_3 < \delta_0$ such that $T_0 \leq T_2$ if and only if $\delta \leq \delta_3$ (Step 013);
- $\delta_1 < \delta_3 < \delta_2$ (Step 014).

Defining $\underline{\delta}_c^{tp} = \delta_1$ and $\overline{\delta}_c^{tp} = \delta_3$ concludes the proof of the proposition. □

I Secret Offers

In this section, we assume that upstream firms offer linear, potentially discriminatory contracts to downstream firms. As mentioned in the paper, we also assume that (i) input suppliers are chosen before downstream competition takes place and (ii) supplier choices are publicly observed. Assumption (i) ensures that, even though upstream offers are secret, a vertically integrated firm knows its upstream market share before setting its downstream price, so that undercutting decisions still trade off the upstream profit effect against the loss of the softening effect. Assumption (ii) rules out implausible situations in which, say, $U_i - D_i$ is expected to supply the entire upstream market at its monopoly upstream price (while its vertically integrated rivals are expected to make no upstream offer); but $U_i - D_i$ finds it profitable to deviate secretly by withdrawing its upstream offer, so that the unintegrated downstream firms end up purchasing from the alternative source of input (or exiting), without this being observed by $U_i - D_i$'s integrated rivals.

The proofs of Lemmas E and F can easily be adapted to show that, given passive beliefs, the equilibrium profit of an unintegrated downstream firm is a decreasing function of the price at which it buys the input, holding fixed the identity of its upstream supplier. This implies that the Bertrand outcome is always an equilibrium, regardless of how many mergers have taken place.

We establish the existence condition for monopoly-like equilibria when $N = M+1$ in Section I.1, and that for collusive-like equilibria when $M = 2$ and $N = 4$ in Section I.2.

I.1 Proof of Proposition 7

Proof. We derive a necessary and sufficient condition for $\Pi(w_m^s, 0) \geq \Pi(w_m^s, 1)$. All calculations are in the Mathematica notebook `09_secret_monopoly.nb`.

We begin by computing equilibrium downstream prices, quantities, and profits in the public offer game (Step 001). We also define δ_{sup} as the threshold above which downstream firms cannot be active when $w = 0$ (Step 002). In the following, we assume $\delta < \delta_{sup}$.

Next, we define the out-of-equilibrium profit functions $\Pi^s(w, w^b)$ and $\Pi^s_d(w, w^b)$ in the private offer game (Step 003). For all w^b, $\Pi^s(w, w^b)$ is concave in w, and the monopoly upstream price under secret offers, w^s_m, is unique (Step 004). $\Pi(w^s_m, 0) - \Pi(w^s_m, 0)$ is concave in δ and positive for $\delta = \delta_{sup}$. Therefore, there exists $\delta^s_m < \delta_{sup}$ such that $\Pi(w^s_m, 1) \leq \Pi(w^s_m, 0)$ if and only if $\delta \geq \delta^s_m$ (Step 005). Finally, we check that δ^s_m is larger than δ_m, the threshold for monopoly-like equilibria in the public offer case (Step 006). □

I.2 Proof of Proposition 8

Proof. All calculations are in the Mathematica notebook `10_secret_collusive.nb`. In the collusive-like equilibrium candidate, both vertically integrated firms offer $w > 0$ to both downstream firms, D_3 purchases from $U_1 - D_1$, and D_4 purchases from $U_2 - D_2$. We compute equilibrium prices and profits in this equilibrium candidate in Step 001 of the Mathematica file. Since beliefs must be correct on the candidate equilibrium path, those prices and profits are the same as under public contracts. This means that vertically integrated firms earn $\Pi(w, 1/2)$ on path. We also redefine δ_{sup} as the cutoff above which downstream firms cannot be active when $w = 0$.

We start with a few preliminary observations. Note that $U_1 - D_1$ can deviate by offering $w - \varepsilon$ to both downstream firms, thereby taking over the entire upstream market. Given passive beliefs, as ε tends to zero, the ensuing equilibrium converges to the equilibrium under public contracts when $U_1 - D_1$ supplies both downstream firms at w. For there to be a collusive-like equilibrium at price w, it must therefore be the case that

$$\Pi\left(w, \frac{1}{2}\right) \geq \Pi(w, 1). \tag{15}$$

Similarly, $U_1 - D_1$ can deviate by withdrawing both of its upstream offers. After such a deviation, both downstream firms purchase from $U_2 - D_2$ at w and the equilibrium downstream prices are as under public contracts. We thus obtain a second non-deviation constraint:

$$\Pi\left(w, \frac{1}{2}\right) \geq \Pi(w, 0). \tag{16}$$

We have shown in the proof of Lemma B that there exist cutoffs $w_1(\delta)$ and $w_2(\delta)$ such that conditions (15)–(16) hold jointly if and only if $w_1(\delta) \leq w \leq w_2(\delta)$. Moreover, $w_1(\delta) \leq w_2(\delta)$ if and only if $\delta \leq \overline{\delta}_c$, $w_1(\delta) = w_2(\delta) = 0$ if $\delta = \overline{\delta}_c$, and $0 < w_1(\delta) <$

$w_2(\delta)$ if $\delta < \overline{\delta}_c$. Hence, there is no symmetric collusive-like equilibrium if $\delta \geq \overline{\delta}_c$, and, regardless of δ, no price outside the interval $[w_1(\delta), w_2(\delta)]$ can be sustained in a symmetric collusive-like equilibrium. In the following, we assume $\delta < \overline{\delta}_c$ and $w \in [w_1(\delta), w_2(\delta)]$

We recompute $\Pi(w, 1)$ and $\Pi(w, 0)$ (Step 002), as well as $w_1(\delta)$, $w_2(\delta)$, and $\overline{\delta}_c$ (Step 003). In Step 004, we also redefine w_m, the monopoly upstream price under public contracts, and δ_2, the threshold above which $w_2(\delta) < w_m$—see the proof of Lemma B. In that step, we also show that $\delta_2 > 0$. Recall from the proof of Lemma B that $\overline{\delta}_c > \delta_2 > \underline{\delta}_c$.

Let $\Pi^U(w_3, w_4, w)$ be $U_1 - D_1$'s profit when it supplies D_3 at w_3 and D_4 at w_4, but $U_2 - D_2$ believes that both downstream firms are supplied at w. We compute this function in Step 005. For there to be a collusive-like equilibrium at price w, it must be the case that $\Pi^U(w_3, w_4, w) \leq \Pi(w, 1/2)$ for every $(w_3, w_4) \in [0, w]^2$. For every $w > 0$, $\Pi^U(w_3, w_4, w)$ is strictly concave in (w_3, w_4) (Step 006). Since that function is also symmetric in (w_3, w_4), the optimal deviation must satisfy $w_3 = w_4$: Thus, let $\Pi^U(w_3, w) \equiv \Pi^U(w_3, w_3, w)$. The function $\Pi^U(\cdot, w)$ is strictly concave (Step 007). Since $\Pi(w, 1/2) \geq \Pi(w, 1) = \Pi^U(w, w)$, this implies that $U_1 - D_1$ does not want to take over the entire upstream market at prices weakly below w if and only if

$$\Phi(w, \delta) \equiv \left. \frac{\partial \Pi^U(w_3, w)}{\partial w_3} \right|_{w_3 = w} \geq 0,$$

a condition we study next.

Φ is strictly decreasing in w; moreover, at $\delta = \overline{\delta}_c$ and $w = w_1(\delta) = w_2(\delta) = 0$, Φ is strictly positive (Step 008). We show that $\Phi(w_2(\delta), \delta)$ is strictly increasing in δ and strictly negative when $\delta = \delta_2$ (Step 009). There therefore exists $\delta_2^U \in (\delta_2, \overline{\delta}_c)$ such that $\Phi(w_2(\delta), \delta) \geq 0$ if and only if $\delta \geq \delta_2^U$. Similarly, $\Phi(w_1(\delta), \delta)$ is strictly increasing in δ and strictly negative when $\delta = \delta_2$ (Step 010), implying the existence of a $\delta_1^U \in (\delta_2, \overline{\delta}_c)$ such that $\Phi(w_1(\delta), \delta) \geq 0$ if and only if $\delta \geq \delta_1^U$. Since Φ is decreasing in w, we have that $\delta_1^U < \delta_2^U$, $\Phi(w, \delta) \geq 0$ for every $\delta \geq \delta_2^U$ and $w \in [w_1(\delta), w_2(\delta)]$, and $\Phi(w, \delta) < 0$ for every $\delta < \delta_1^U$ and $w \in [w_1(\delta), w_2(\delta)]$. If $\delta \in [\delta_1^U, \delta_U^2)$, then there exists $w^U(\delta) \in [w_1(\delta), w_2(\delta))$ (defined in Step 011) such that $\Phi(w, \delta) \geq 0$ if and only if $w \in [w_1(\delta), w^U(\delta)]$. Thus, there is no profitable deviation in which $U_1 - D_1$ takes over the entire upstream market if and only if $\delta \geq \delta_2^U$, or $\delta \in [\delta_1^U, \delta_2^U)$ and $w \in [w_1(\delta), w^U(\delta)]$.

Let $\Pi^u(w_3, w)$ be $U_1 - D_1$'s profit when it supplies D_3 at w_3, but $U_2 - D_2$ and D_4 believe that D_3 is supplied at w. We compute this function in Step 012 and show that it is strictly concave in w_3 in Step 013. For there to be a collusive-like equilibrium at price w, it must be the case that $\Pi^u(w_3, w) \leq \Pi(w, 1/2)$ for every $w_3 \in [0, w]$. Since $\Pi^u(w, w) = \Pi(w, 1/2)$ and by concavity of $\Pi^u(\cdot, w)$, this condition is satisfied if and only if

$$\Psi(w, \delta) \equiv \left. \frac{\partial \Pi^u(w_3, w)}{\partial w_3} \right|_{w_3 = w} \geq 0.$$

We now argue that this condition is satisfied whenever $\delta \in [\delta_2^U, \overline{\delta}_c)$ and $w \in [w_1(\delta), w_2(\delta)]$, or $\delta \in [\delta_1^U, \delta_2^U)$ and $w \in [w_1(\delta), w_U(\delta)]$. Ψ is strictly decreasing in w (Step 014). Moreover, $\Psi(w_2(\delta), \delta)$ is strictly increasing in δ and strictly positive when $\delta = \delta_2^U$ (Step 015). By monotonicity of Ψ in w, it follows that $\Psi(w, \delta) \geq 0$ when $\delta \in [\delta_2^U, \overline{\delta}_c)$ and $w \in [w_1(\delta), w_2(\delta)]$. Similarly, $\Psi(w^U(\delta), \delta)$ is strictly increasing in δ and strictly positive when $\delta = \delta_1^U$. By monotonicity of Ψ in w, it follows that $\Psi(w, \delta) \geq 0$ when $\delta \in [\delta_1^U, \delta_2^U)$ and $w \in [w_1(\delta), w_U(\delta)]$.

Since we have exhausted all possible deviations, we can conclude. There exist symmetric collusive-like equilibria in passive beliefs if and only if $\delta \in [\delta_1^U, \overline{\delta}_c)$. If $\delta \in [\delta_2^U, \overline{\delta}_c)$, then the set of input prices that can be sustained in such an equilibrium is $[w_1(\delta), w_2(\delta)]$—the same set as under public contracts (see the proof of Lemma B). If $\delta \in [\delta_1^U, \delta_2^U)$, then the set of input prices that can be sustained in such an equilibrium is $[w_1(\delta), w^U(\delta)]$; this set is a strict subset of $[w_1(\delta), w_2(\delta)]$, the set of prices that can be sustained under public contracts. Setting $\underline{\delta}_c^s \equiv \delta_1^U$ and noting that $\delta_1^U > \delta_2 > \underline{\delta}_c$ concludes the proof. $\square$