B Supplementary Appendix (not for publication)

This supplementary appendix provides ommitted proofs and establishes results to complement the main analysis.

B.1 Omitted proofs

B.1.1 Proof of Lemma 3

Twice continuous differentiability follows directly from the fact that $D(\pi,\xi)$, $\mu_{h,t}$ and $\pi_{t,u}$ are all twice continuously differentiable. Boundedness is proved from the following direct calculations. First, note that:

$$\pi_{t,u} = \frac{\mu_{h,u}-\mu_{h,t}}{1-\mu_{h,t}} = 1 - \frac{1-\mu_{h,u}}{1-\mu_{h,t}},$$

which implies that:

$$\frac{\partial \pi_{t,u}}{\partial u} = \frac{\mu'_{h,u}}{1-\mu_{h,t}} \quad \text{and} \quad \frac{\partial \pi_{t,u}}{\partial t} = -\mu'_{h,t}\frac{1-\mu_{h,u}}{[1-\mu_{h,t}]^2}. \tag{50}$$

<u>First derivative with respect to t: $\mathcal{D}_t(t,u,\xi)$.</u> Then we can calculate the partial derivative of $\mathcal{D}(t,u,\xi)$ with respect to t

$$\begin{aligned}\mathcal{D}_t(t,u,\xi) &= \mu'_{h,t}D(1,\xi) - \mu'_{h,t}D(\pi_{t,u},\xi) + \left[1-\mu_{h,t}\right] D_\pi(\pi_{t,u},\xi)\frac{\partial \pi_{t,u}}{\partial t} \\ &= \mu'_{h,t}\left\{D(1,\xi) - D(\pi_{t,u},\xi) - \frac{1-\mu_{h,u}}{1-\mu_{h,t}}D_\pi(\pi_{t,u},\xi)\right\},\end{aligned} \tag{51}$$

which is bounded over the relevant range, $\Delta \times [\underline{\xi},\overline{\xi}]$, because: $\mu'_{h,t} = \gamma e^{-\gamma t}$ is bounded; $(1-\mu_{h,u})/(1-\mu_{h,t}) = e^{-\gamma(u-t)}$ and so is bounded; $\pi_{t,u} \in [0,1]$ and so is bounded; $D(\pi,\xi)$ and $D_\pi(\pi,\xi)$ are continuous over the compact $[0,1]\times[\underline{\xi},\overline{\xi}]$ and so are bounded as well.

<u>First derivative with respect to u: $\mathcal{D}_u(t,u,\xi)$.</u> We have:

$$\mathcal{D}_u(t,u,\xi) = \left[1-\mu_{h,t}\right] D_\pi(\pi_{t,u},\xi)\frac{\partial \pi_{t,u}}{\partial u} = \mu'_{h,u}D_\pi(\pi_{t,u},\xi),$$

which is bounded over the relevant range for the same reasons as above.

<u>First derivative with respect to ξ: $\mathcal{D}_\xi(t,u,\xi)$.</u> We have

$$\mathcal{D}(t,u,\xi) = \mu_{h,t}D_\xi(1,\xi) + \left[1-\mu_{h,t}\right] D_\xi(\pi_{t,u},\xi), \tag{52}$$

which is bounded over the relevant range for the same reasons as above.

Second derivative with respect to (t,t). It is equal to:

$$\begin{aligned}\mathcal{D}_{t,t}(t,u,\xi) =& \mu''_{h,t}\left\{D(\pi,\xi) - D(\pi_{t,u},\xi) - \frac{1-\mu_{h,t}}{1-\mu_{h,u}} D_\pi(\pi_{t,u},\xi)\right\} \\ &+ \mu'_{h,t}\frac{\mu'_{h,t}}{1-\mu_{h,t}}\left[\frac{1-\mu_{h,u}}{1-\mu_{h,t}}\right]^2 D_{\pi\pi}(\pi_{t,u},\xi),\end{aligned}$$

which is bounded over the relevant range for the same reason as above and after noting that $\mu'_h(t)/\left[1-\mu_h(t)\right] = \gamma$.

Second derivative with respect to (t,u). It is equal to:

$$\mathcal{D}_{t,u}(t,u,\xi) = -\mu'_{h,t}\mu'_{h,u}\frac{1-\mu_{h,u}}{\left[1-\mu_{h,t}\right]^2} D_{\pi,\pi}(\pi_{t,u},\xi),$$

which is bounded over the relevant range for the same reasons as above.

Second derivative with respect to (t,ξ). It is equal to:

$$\mathcal{D}_{t,\xi}(t,u,\xi) = \mu'_{h,t}\left\{D_\xi(1,\xi) - D_\xi(\pi_{t,u},\xi) - \frac{1-\mu_{h,u}}{1-\mu_{h,t}} D_{\pi,\xi}(\pi_{t,u},\xi)\right\},$$

which is bounded over the relevant range for the same reasons as above.

Second derivative with respect to (u,u): $\mathcal{D}_{u,u}(t,u,\xi)$.

$$\mathcal{D}_{u,u}(t,u,\xi) = \mu''_{h,u} D_\pi(\pi_{t,u},\xi) + \mu'_{h,u}\frac{\mu'_{h,u}}{1-\mu_{h,t}} D_{\pi,\pi}(\pi_{t,u},\xi).$$

which is bounded over the relevant range since $\mu'_{h,u}/\left[1-\mu_{h,t}\right] = \gamma e^{-\gamma(u-t)}$.

Second derivative with respect to (u,ξ): $\mathcal{D}_{u,\xi}(t,u,\xi)$.

$$\mathcal{D}_{u,\xi}(t,u,\xi) = \mu'_{h,u} D_{\pi,\xi}(\pi_{t,u},\xi),$$

which is bounded over the relevant range.

Second derivatives with respect to (ξ,ξ): $\mathcal{D}_{\xi,\xi}(t,u,\xi)$.

$$\mathcal{D}_{\xi,\xi} = \mu_{h,t} D_{\xi,\xi}(\theta,\xi) + \left[1-\mu_{h,t}\right] D_{\xi,\xi}(\pi_{t,u},\xi),$$

which is bounded over the relevant range.

$\mathcal{D}_\xi(t,u,\xi)$ is bounded away from zero. This follows directly from the formula for $\mathcal{D}_\xi(t,u,\xi)$ because, on the one hand, $\mu_{h,t} \in [0,1]$ and, on the other hand, $D_\xi(\pi,\xi)$ is continuous and thus bounded away from zero over the compact $[0,1] \times [\underline{\xi},\overline{\xi}]$.

B.1.2 Proof of Lemma 6

First, note that, by an application of the Implicit Function Theorem, the holding cost $\xi_u(\rho)$ is continuously differentiable, with a derivative that can be written

$$\frac{d\xi_u(\rho)}{du} = -\frac{A_u(\rho)+B_u(\rho)}{C_u(\rho)},$$

$$\text{where } A_u(\rho) = \rho\mathcal{D}(u,u,\xi_u(\rho)); \quad B_u(\rho) = e^{-\rho u}\mathcal{D}_u(0,u,\xi_u(\rho)) + \int_0^u \rho e^{-\rho(u-t)}\mathcal{D}_u(t,u,\xi_u(\rho))\,dt;$$
$$C_u(\rho) = e^{-\rho u}\mathcal{D}_\xi(0,u,\xi_u(\rho)) + \int_0^u \rho e^{-\rho(u-t)}\mathcal{D}_\xi(t,u,\xi_u(\rho))\,dt.$$

To obtain the asymptotic behavior of $A_u(\rho)$, we apply a second-order Taylor formula:

$$\begin{aligned}
A_u(\rho) =& \rho\left\{\mathcal{D}(u,u,\xi_u^\star) + \mathcal{D}_\xi(u,u,\xi_u^\star)\left[\xi_u-\xi_u^\star\right] + \frac{\mathcal{D}_{\xi\xi}(u,u,\hat{\xi}_u(\rho))}{2}\left[\xi_u-\xi_u^\star\right]^2\right\} \\
=& \rho\Bigg\{\mathcal{D}_\xi(u,u,\xi_u^\star)\left[\frac{1}{\rho}\frac{\mathcal{D}_t(u,u,\xi_u^\star)}{\mathcal{D}_\xi(u,u,\xi_u^\star)} + o_\alpha\left(\frac{1}{\rho}\right)\right] \\
&\qquad + \frac{\mathcal{D}_{\xi\xi}(u,u,\hat{\xi}_u(\rho))}{2}\left[\frac{1}{\rho}\frac{\mathcal{D}_t(u,u,\xi_u^\star)}{\mathcal{D}_\xi(u,u,\xi_u^\star)} + o_\alpha\left(\frac{1}{\rho}\right)\right]^2\Bigg\} \\
=& \mathcal{D}_t(u,u,\xi_u^\star) + o_\alpha(1),
\end{aligned}$$

where the second line follows after noting that $\mathcal{D}(u,u,\xi_u^\star)=0$ by definition of $\xi_u^\star$ and after plugging in the approximation of Proposition 5.

Turning to $B_u(\rho)$, we first integrate by part to note that:

$$\begin{aligned}
B_u(\rho) &= \mathcal{D}_u(u,u,\xi_u(\rho)) - \frac{1}{\rho}\int_0^u \rho e^{-\rho(u-t)}\mathcal{D}_{u,t}(t,u,\xi_u(\rho))\,dt \\
&= \mathcal{D}_u(u,u,\xi_u(\rho)) + o_\alpha(1),
\end{aligned}$$

since, by Lemma 3, $\mathcal{D}(t,u,\xi)$ has bounded first and second derivatives. Given that $\mathcal{D}_u(t,u,\xi)$ has bounded first derivatives, it is uniformly continuous over $\Delta\times[\underline{\xi},\overline{\xi}]$. Together with the fact that $\xi_u(\rho)=\xi_u^\star+o_\alpha(1)$, this implies that:

$$B_u(\rho) = \mathcal{D}_u(u,u,\xi_u^\star) + o_\alpha(1).$$

The same arguments applied to $C_u(\rho)$ show that:

$$C_u(\rho) = \mathcal{D}_\xi(u,u,\xi_u^\star) + o_\alpha(1).$$

Taken together, we obtain that

$$\frac{d\xi_u(\rho)}{du} = -\frac{\mathcal{D}_t(u,u,\xi_u^\star) + \mathcal{D}_u(u,u,\xi_u^\star)}{\mathcal{D}_\xi(u,u,\xi_u^\star)} + o_\alpha(1) = \frac{d\xi_u^\star}{du} + o_\alpha(1).$$

B.1.3 Proof of Lemma 7

Since $D(\pi,\xi)$ is uniformly continuous over $[0,1]\times[\underline{\xi},\overline{\xi}]$, and since $\xi_u(\rho) = \xi_u^\star + o_\alpha(1)$, it follows that $q_{\ell,t,u} = q_{\ell,t,u}^\star + o_\alpha(1)$, and $q_{h,u} = q_{h,u}^\star + o_\alpha(1)$. Using the same argument we obtain that:

$$\frac{\partial q_{\ell,t,u}}{\partial t} = D_\pi(\pi_{t,u},\xi_u(\rho))\frac{\partial \pi_{t,u}}{\partial t} = D_\pi(\pi_{t,u},\xi_u^\star)\frac{\partial \pi_{t,u}}{\partial t} + o_\alpha(1),$$

Next:

$$\begin{aligned}\frac{\partial q_{\ell,t,u}}{\partial u} &= D_\pi(\pi_{t,u},\xi_u(\rho))\frac{\partial \pi_{t,u}}{\partial u} + D_\pi(\pi_{t,u},\xi_u(\rho))\frac{d\xi_u(\rho)}{du}\\ &= D_\pi(\pi_{t,u},\xi_u^\star)\frac{\partial \pi_{t,u}}{\partial u} + D_\xi(\pi_{t,u},\xi_u^\star)\frac{d\xi_u^\star}{du} + o_\alpha(1),\end{aligned}$$

using the same argument as above as well as Lemma 6. Lastly,

$$\frac{dq_{h,u}}{du} = D_\pi(1,\xi_u(\rho))\frac{d\xi_u(\rho)}{du} = D_\xi(1,\xi_u^\star)\frac{d\xi_u^\star}{du} + o(1),$$

using the same argument as above.

B.1.4 Proof of Lemma 8

One easily verifies that $m(0,\varepsilon) = 0$ and $\lim_{q\to\infty} m(q,\varepsilon) = 1$. Clearly, $m(q,\varepsilon)$ is continuous over $(q,\varepsilon)\in[0,\infty)\times(0,\infty)$, so the only potential difficulty lies in proving continuity at all points $(q,0)$. For this consider $q \geq 0$ and a sequence $(q_n,\varepsilon_n)\to(q,0)$. We need to show that $m(q_n,\varepsilon_n)\to\min\{q,1\}$. If $q\geq 1$, the numerator of $1-m(q_n,\varepsilon_n)$ is positive and bounded above by $\ln(1+e^{\frac{-1}{1-\varepsilon}})$ which goes to $\ln(1+e^{-1})$, and the denominator goes to $+\infty$. Therefore, $1-m(q_n,\varepsilon_n)\to 0$ and so $m(q_n,\varepsilon_n)\to 1$. Consider now $q<1$. For $x>0$, let $\phi(x)\equiv x\ln\left(1+e^{\frac{1}{x}}\right)$ and let $\phi(0) = \lim_{x\to 0^+}\phi(x) = 1$, so that $\phi(x)$ is extended by continuity at 0^+. We can then write:

$$1-m(q,\varepsilon) = \vartheta(q,\varepsilon)\frac{\phi\circ\psi(q,\varepsilon)}{\phi(\varepsilon)}, \text{ where } \vartheta(q,\varepsilon)\equiv\left(1-\frac{q^{1-\varepsilon}}{1-\varepsilon}\right) \text{ and } \psi(q,\varepsilon)\equiv\varepsilon\left(1-\frac{q^{1-\varepsilon}}{1-\varepsilon}\right)^{-1}.$$

Clearly, for $q<1$ both $\vartheta(q,\varepsilon)$ and $\psi(q,\varepsilon)$ are continuous at $(q,0)$, with $\vartheta(q,0) = 1-q$ and $\psi(q,0) = 0$. The function $\phi(x)$ is continuous at 0 by construction. It then follows that

$$1-m(q_n,\varepsilon_n)\to 1-m(q,0) = 1-q.$$

Next, consider the first and second derivatives of $m(q,\varepsilon)$:

$$m(q,\varepsilon) = 1 - \frac{\ln\left(1+e^{\frac{1}{\varepsilon}\left(1-\frac{q^{1-\varepsilon}}{1-\varepsilon}\right)}\right)}{\ln\left(1+e^{\frac{1}{\varepsilon}}\right)} \tag{53}$$

$$m_q(q,\varepsilon) = \frac{1}{\varepsilon\ln\left(1+e^{\frac{1}{\varepsilon}}\right)} q^{-\varepsilon} \frac{e^{\frac{1}{\varepsilon}\left(1-\frac{q^{1-\varepsilon}}{1-\varepsilon}\right)}}{1+e^{\frac{1}{\varepsilon}\left(1-\frac{q^{1-\varepsilon}}{1-\varepsilon}\right)}} > 0 \tag{54}$$

$$m_{qq}(q,\varepsilon) = \frac{-1}{\varepsilon\ln\left(1+e^{\frac{1}{\varepsilon}}\right)} \left[\varepsilon q^{-(1+\varepsilon)} \frac{e^{\frac{1}{\varepsilon}\left(1-\frac{q^{1-\varepsilon}}{1-\varepsilon}\right)}}{1+e^{\frac{1}{\varepsilon}\left(1-\frac{q^{1-\varepsilon}}{1-\varepsilon}\right)}} + \frac{q^{-2\varepsilon}}{\varepsilon} \frac{e^{\frac{1}{\varepsilon}\left(1-\frac{q^{1-\varepsilon}}{1-\varepsilon}\right)}}{\left(1+e^{\frac{1}{\varepsilon}\left(1-\frac{q^{1-\varepsilon}}{1-\varepsilon}\right)}\right)^2}\right] < 0 \tag{55}$$

Clearly, $m(q,\varepsilon)$ is increasing and concave, and three times continuously differentiable over $(q,\varepsilon) \in [0,\infty)\times(0,\infty)$. The limits of the first and second derivative follow from similar arguments as above.

B.1.5 Proof of Lemma 10

The continuity of $\overline{Q}_u$ is obvious. That $\overline{Q}_0 = (s-\mu_{h,0})/(1-\mu_{h,0})^{1+1/\sigma}$ follows from the definition of $\overline{Q}_u$, and $\overline{Q}_{T_f} = 0$ follows by definition of T_f. Next, after taking derivatives with respect to u we find that $\text{sign}\left[\overline{Q}'_u\right] = \text{sign}\left[F_u\right]$, where:

$$\begin{aligned} F_u \equiv & (s-\mu_{h,u})\left[(1-\mu_{h,0})^{1+1/\sigma} + \int_0^u \rho e^{\rho t}(1-\mu_{h,t})^{1+1/\sigma} dt\right] \\ & - (1-\mu_{h,u})^{1+1/\sigma}\left[s-\mu_{h,0} + \int_0^u \rho e^{\rho t}(s-\mu_{h,t})dt\right], \end{aligned} \tag{56}$$

is continuously differentiable. Taking derivatives once more, we find that $\text{sign}\left[F'_u\right] = \text{sign}\left[G_u\right]$ where:

$$\begin{aligned} G_u \equiv & -\left[(1-\mu_{h,0})^{1+1/\sigma} + \int_0^u \rho e^{\rho t}(1-\mu_{h,t})^{1+1/\sigma} dt\right] \\ & + \left(1+\frac{1}{\sigma}\right)(1-\mu_{h,u})^{1/\sigma}\left[s-\mu_{h,0} + \int_0^u \rho e^{\rho t}(s-\mu_{h,t})dt\right], \end{aligned} \tag{57}$$

is continuously differentiable. Now suppose that $\overline{Q}'_u = 0$. Then $F_u = 0$ and, after substituting (56) into (57):

$$G_u = \left[-\frac{(1-\mu_{h,u})^{1+1/\sigma}}{s-\mu_{h,u}} + \left(1+\frac{1}{\sigma}\right)(1-\mu_{h,u})^{1/\sigma}\right]\left[s-\mu_{h,0} + \int_0^u \rho e^{\rho t}(s-\mu_{h,t})dt\right]. \tag{58}$$

Thus,

R1. *Suppose that $F_u = 0$ for some $u \in [0,T_f)$. Then sign* $[F'_u] =$ *sign* $\left[\frac{s-\mu_{h,u}}{1-\mu_{h,u}} - \frac{\sigma}{1+\sigma}\right]$.

Now note that $G_0 = (1-\mu_{h,0})^{1+1/\sigma}\left(1+\frac{1}{\sigma}\right)\left(-\frac{\sigma}{1+\sigma}+\frac{s-\mu_{h,0}}{1-\mu_{h,0}}\right)$. Thus,

R 2. *If $\frac{s-\mu_{h,0}}{1-\mu_{h,0}} \leq \frac{\sigma}{1+\sigma}$, then $F_u < 0$ for all $u > 0$.*

To see this, first note that, from application of the Mean Value Theorem (see, e.g., Theorem 5.11 in Apostol, 1974), it follows that $F_u < 0$ for small u. Indeed, since $F_0 = 0$, $F_u = uF'_v$, for some $v \in (0,u)$. But sign$[F'_v] =$ sign$[G_v]$. Now, since $G_0 \leq 0$, G_v is negative as long as u is small enough. But if F_u is negative for small u, it has to stay negative for all u. Otherwise, it would need to cross the x-axis from below at some $u > 0$, which is impossible given Result R1 and the assumption that $\frac{s-\mu_{h,0}}{1-\mu_{h,0}} \leq \frac{\sigma}{1+\sigma}$.

R 3. *If $\frac{s-\mu_{h,0}}{1-\mu_{h,0}} > \frac{\sigma}{1+\sigma}$, then $F_u > 0$ for small u and F_u changes sign only once in the interval $(0, T_f)$.*

$F_u > 0$ for small u follows from applying the same reasoning as in the above paragraph, since when $\frac{s-\mu_{h,0}}{1-\mu_{h,0}} > \frac{\sigma}{1+\sigma}$ we have $G_0 > 0$. Since $F_{T_s} < 0$, then F_u must cross zero at least once between 0 and T_s. The first time F_u crosses zero, it must be from above and hence with a negative slope. Thus, the expression in Result R1 for the sign of F'_u when $F_u = 0$ is negative and this expression is decreasing in u. Therefore, F_u cannot cross zero from below at a later time and hence it only crosses zero once. $\square$

B.1.6 Proof of Lemma 12

To prove that Q_u is continuously differentiable except in T_1 and T_2, we apply the Implicit Function Theorem (see, e.g., Theorem 13.7 in Apostol, 1974). We note that (40) writes $K(u, Q_u) = 0$, where

$$K(u,Q) \equiv \Big((1-\mu_{h,0})\min\{(1-\mu_{h,0})^{1/\sigma}Q, 1\} + \mu_{h,0} - s\Big) + \int_0^u \rho e^{\rho t}\Big((1-\mu_{h,t})\min\{(1-\mu_{h,t})^{1/\sigma}Q, 1\} + \mu_{h,t} - s\Big)\, dt. \tag{59}$$

We consider first the case where $\overline{Q}_u > (1-\mu_{h,0})^{-1/\sigma}$ for some u. Recall that we defined $0 < T_1 < T_2 < T_f$ such that $\overline{Q}_{T_1} = \overline{Q}_{T_2} = (1-\mu_{h,0})^{-1/\sigma}$. Since $(1-\mu_{h,0})^{1/\sigma}Q_u < 1$ for $u < T_1$, we restrict attention to the domain $\{(u,Q) \in \mathbb{R}^2_+ : u < T_1 \text{ and } Q < (1-\mu_{h,0})^{-1/\sigma}\}$. In this domain, equation (59) can be written

$$K(u,Q) = \Big((1-\mu_{h,0})^{1+1/\sigma}Q + \mu_{h,0} - s\Big) + \int_0^u \rho e^{\rho t}\Big((1-\mu_{h,t})^{1+1/\sigma}Q + \mu_{h,t} - s\Big)\, dt.$$

To apply the Implicit Function Theorem, we need to show that $K(u,Q)$ is continuously differentiable. To see this, first note that the partial derivative of $K(u,Q)$ with respect to u is

$$\frac{\partial K}{\partial u} = \rho e^{\rho u}\Big((1-\mu_{h,u})^{1+1/\sigma}Q + \mu_{h,u} - s\Big)$$

and is continuous. The partial derivative with respect to Q is

$$\frac{\partial K}{\partial Q} = (1-\mu_{h,0})^{1+1/\sigma} + \int_0^u \rho e^{\rho t}\left(1-\mu_{h,t}\right)^{1+1/\sigma}\, dt$$

and is continuous and strictly positive. Therefore, we can apply the Implicit Function Theorem and state that

$$Q'_u = -\frac{\partial K/\partial u}{\partial K/\partial Q} = \frac{\rho e^{\rho u}\left(s-\mu_{h,u}-(1-\mu_{h,u})^{1+1/\sigma}Q\right)}{(1-\mu_{h,0})^{1+1/\sigma} + \int_0^u \rho e^{\rho t}\left(1-\mu_{h,t}\right)^{1+1/\sigma}\, dt}.$$

The same reasoning and expression for Q'_u obtain for $u > T_2$, as well as for all u in the case where $\overline{Q}_u \leq (1-\mu_{h,0})^{-1/\sigma}$ for all u.

The second domain to consider is $\{(u,Q)\in\mathbb{R}^2_+ : T_1 < u < T_2 \text{ and } Q > (1-\mu_{h,0})^{-1/\sigma}\}$. In this domain, equation (59) can be written, using the definition of $\Psi(Q)$,

$$K(u,Q) = (1-s) + \int_0^{\Psi(Q)} \rho e^{\rho t}(1-s)\, dt + \int_{\Psi(Q)}^u \rho e^{\rho t}\left((1-\mu_{h,t})^{1+1/\sigma}Q + \mu_{h,t} - s\right) dt.$$

The partial derivative of $K(u,Q)$ with respect to u is

$$\frac{\partial K}{\partial u} = \rho e^{\rho u}\left((1-\mu_{h,u})^{1+1/\sigma}Q + \mu_{h,u} - s\right)$$

and is continuous. Noting that $\Psi(Q)$ is differentiable, the partial derivative with respect to Q is

$$\frac{\partial K}{\partial Q} = \int_{\Psi(Q)}^u \rho e^{\rho t}\left(1-\mu_{h,t}\right)^{1+1/\sigma}\, dt$$

and is continuous since $\Psi(Q)$ is continuous. Moreover, since $\Psi(Q) < u$ in its domain, then $\partial K/\partial Q > 0$. Therefore, we can apply the Implicit Function Theorem and

$$Q'_u = -\frac{\partial K/\partial u}{\partial K/\partial Q} = \frac{\rho e^{\rho u}\left(s-\mu_{h,u}-(1-\mu_{h,u})^{1+1/\sigma}Q\right)}{\int_{\psi_u}^u \rho e^{\rho t}\left(1-\mu_{h,t}\right)^{1+1/\sigma}\, dt},$$

where we used that $\psi_u \equiv \Psi(Q_u)$. □

B.1.7 Proof of Lemma 13

For $u \in (T_1, T_2)$, we have $Q_u \neq \overline{Q}_u$ and therefore and therefore $\Psi(Q_u) = \psi_u > 0$. By definition of ψ_u, we also have

$$Q_u = (1-\mu_{h,\psi_u})^{-1/\sigma}. \tag{60}$$

Replacing into equation (42) for Q'_u of Lemma 12 , one obtains that:

$$\text{sign}\left[Q'_u\right] = \text{sign}\left[X_u\right] \text{ where } X_u \equiv s - \mu_{h,u} - (1-\mu_{h,u})\left(\frac{1-\mu_{h,u}}{1-\mu_{h,\psi_u}}\right)^{1/\sigma}.$$

As noted above, Q_u and thus X_u changes sign at least once over (T_1, T_2). Now, for any u_0 such that $X_{u_0} = 0$, we have $Q'_{u_0} = 0$ and, given (60), $\psi'_{u_0} = 0$. Taking the derivative of X_u at such u_0, and using $X_{u_0} = 0$, leads:

$$\text{sign}\left[X'_{u_0}\right] = \text{sign}\left[-1 + \left(1+\frac{1}{\sigma}\right)\left(\frac{1-\mu_{h,u_0}}{1-\mu_{h,\psi_{u_0}}}\right)^{1/\sigma}\right] = \text{sign}\left[Y_{u_0}\right],$$
$$\text{where } Y_u \equiv -1 + \left(1+\frac{1}{\sigma}\right)\frac{s-\mu_{h,u}}{1-\mu_{h,u}},$$

where the second equality follows by using $X_{u_0} = 0$. Now take u_0 to be the *first* time X_u changes sign during (T_1, T_2). Since $X_{u_0} = 0$, X_u strictly positive to the left of u_0, and X_u strictly negative to the right of u_0, we must have that $X'_{u_0} \leq 0$. Suppose, then, that X_u changes sign once more during (T_1, T_2) at some time u_1. The same reasoning as before implies that, at u_1, $X'_{u_1} \geq 0$. But this is impossible since Y_u is strictly decreasing. □

B.1.8 Proof of Lemma 14

Proof of the limit of $T_f(\rho)$, in equation (43). Recall that $T_f(\rho)$ solves $\mathbb{E}\left[\mu_{h,\tau_u}\right] = s$ and that $T_f(\rho) \geq T_s$. Note also that $\Pr(\tau_u \leq t) = \min\{e^{-\rho(u-t)}, 1\}$. Therefore, a increases in u and ρ induce first-order stochastic dominance shift. Since $\mu_{h,t}$ is increasing, it follows that $\mathbb{E}\left[\mu_{h,\tau_u}\right]$ is strictly increasing in u and ρ, and therefore that $T_f(\rho)$ is strictly decreasing in ρ. Thus, $T_f(\rho)$ admits a limit $T_f(\infty)$ as $\rho \to \infty$. Since $T_f(\rho)$ is greater than the limit, and since $\mathbb{E}\left[\mu_{h,\tau_u}\right]$ is increasing in u, we have: $\mathbb{E}\left[\mu_{h,\tau_u}\right] \leq s$ for $u = T_f(\infty)$. Taking the limit as $\rho \to \infty$ we find that $\mu_{h,T_f(\infty)} \leq s$ so that $T_f(\infty) \leq T_s$. Since $T_f(\rho) \geq T_s$, the result follows. □

Proof of the first–order expansion, in equation (44). Let

$$f(t,\rho) \equiv (1-\mu_{h,t})\min\left\{(1-\mu_{h,t})^{1/\sigma}Q_u(\rho), 1\right\} + \mu_{h,t} - s. \tag{61}$$

By its definition, $Q_u(\rho)$ solves: $\mathbb{E}\left[f(\tau_u, \rho)\right]$. Note that, for each ρ, $f(t,\rho)$ is continuously differentiable with respect to t except at $t = \psi_u(\rho)$ such that $(1-\mu_{h\psi(\rho)})^{1/\sigma}Q_u(\rho) = 1$. Thus, we can integrate the above by part and obtain:

$$0 = \int_0^u \rho e^{-\rho(u-t)} f(t,\rho)\, dt = f(u,\rho) - \int_0^u e^{-\rho(u-t)} f_t(t,\rho)\, dt, \tag{62}$$

where $f_t(t,\rho)$ denotes the partial derivative of $f(t,\rho)$ with respect to t. Now consider a sequence of ρ going to infinity and the associated sequence of $Q_u(\rho)$. Because $Q_u(\rho)$ is bounded above by $(1-\mu_{h,u})^{-1/\sigma}$, this sequence has at least one accumulation point $Q_u(\infty)$. Taking the limit in (62) along a subsequence converging to this accumulation point, we obtain that $Q_u(\infty)$ solves the equation

$$(1-\mu_{h,u})\min\{(1-\mu_{h,u})^{1/\sigma}Q_u(\infty),1\}+\mu_{h,u}-s=0.$$

whose unique solution is $Q_u(\infty)=(s-\mu_{h,u})/(1-\mu_{h,u})^{1+1/\sigma}$. Thus $Q_u(\rho)$ has a unique accumulation point, and therefore converges towards it. To obtain the asymptotic expansion, we proceed with an additional integration by part in equation (62):

$$\begin{aligned}0=&f(u,\rho)+\frac{1}{\rho}f_t(0,\rho)e^{-\rho u}+\frac{1}{\rho}\int_0^u f_{tt}(t,\rho)e^{-\rho(u-t)}\,dt\\&+\frac{1}{\rho}e^{-\rho(u-\psi_u(\rho))}\left[f_t(\psi_u(\rho)^+,\rho)-f_t(\psi_u(\rho)^-,\rho)\right].\end{aligned}$$

where the term on the second line arises because f_t is discontinuous at $\psi_u(\rho)$. Given that $Q_u(\rho)$ converges and is therefore bounded, the third, fourth and fifth terms on the first line are $o(1/\rho)$. For the second line we note that, since $Q_u(\rho)$ converges to $Q_u(\infty)$, $\psi_u(\rho)$ converges to $\psi_u(\infty)$ such that $(1-\mu_{h\psi_u(\infty)})^{1/\sigma}Q_u(\infty)=1$. In particular, one easily verifies that $\psi_u(\infty)<u$. Therefore $e^{-\rho(u-\psi_u(\rho))}$ goes to zero as $\rho\to\infty$, so the term on the second line is also $o(1/\rho)$. Taken together, this gives:

$$0=f(u,\rho)-\frac{1}{\rho}f_t(u,\rho)+o\left(\frac{1}{\rho}\right). \tag{63}$$

Equation (44) obtains after substituting in the expressions for $f(u,\rho)$ and $f_t(u,\rho)$, using that $\mu'_{h,t}=\gamma(1-\mu_{h,t})$. □

Proof of the convergence of the argmax, in equation (45). First one easily verify that $Q_u(\infty)$ is hump–shaped (strictly decreasing) if and only if $Q_u(\rho)$ is hump–shaped (strictly decreasing). So if $\frac{s-\mu_{h,0}}{1-\mu_{h,0}}\leq\frac{\sigma}{1+\sigma}$, then both $Q_u(\rho)$ and $Q_u(\infty)$ are strictly decreasing, achieve their maximum at $u=0$, and the result follows. Otherwise, if $\frac{s-\mu_{h,0}}{1-\mu_{h,0}}>\frac{\sigma}{1+\sigma}$, consider any sequence of ρ going to infinity and the associated sequence of $T_\psi(\rho)$. Since $T_\psi(\rho)<T_f(\rho)<T_f(0)$, the sequence of $T_{\psi(\rho)}$ is bounded and, therefore, it has at least one accumulation point, $T_{\psi(\infty)}$. At each point along the sequence, $T_\psi(\rho)$ maximizes $Q_u(\rho)$. Using equation (42) to write the corresponding first–order condition, $Q'_{T_\psi(\rho)}=0$, we obtain after rearranging that

$$Q_{T_\psi(\rho)}(\rho)=\frac{s-\mu_{h,T_\psi(\rho)}}{1-\mu_{h,T_\psi(\rho)}}=Q_{T_\psi(\rho)}(\infty)\geq Q_{T_\psi^*}(\rho).$$

where T_ψ^* denotes the unique maximizer of $Q_u(\infty)$. Letting ρ go to infinity on both sides of the equation, we find

$$Q_{T_\psi(\infty)}(\infty)\geq Q_{T_\psi^*}(\infty).$$

But since T_ψ^* is the unique maximizer of $Q_u(\infty)$, $T_\psi(\infty) = T_\psi^*$. Therefore, $T_\psi(\rho)$ has a unique accumulation point, and converges towards it. $\square$

B.1.9 Proof of Lemma 15

Given that $\Delta_u = (1-\mu_{h,u})^{1/\sigma} Q_u$, we have

$$\Delta'_u = -\frac{\gamma}{\sigma}(1-\mu_{h,u})^{1/\sigma} Q_u + (1-\mu_{h,u})^{1/\sigma} Q'_u.$$

Using the formula (42) for Q'_u, in Lemma 12, we obtain:

$$\begin{aligned}
\operatorname{sign}\left[\Delta'_u\right] &= \operatorname{sign}\left[-\frac{\gamma}{\sigma}Q_u + Q'_u\right] \\
= \operatorname{sign}\Bigg[&- \frac{\gamma}{\sigma} Q_u \left(\mathbb{I}_{\{\psi_u = 0\}}(1-\mu_{h,0})^{1+1/\sigma} + \int_{\psi_u}^{u} \rho e^{\rho t}(1-\mu_{h,t})^{1+1/\sigma}\, dt \right) \\
&+ \rho e^{\rho u}\left(s - \mu_{h,u} - (1-\mu_{h,u})^{1+1/\sigma} Q_u \right) \Bigg].
\end{aligned} \tag{64}$$

We first show:

R 4. $\Delta'_u < 0$ *for* u *close to zero.*

To show this result, first note that when u is close to zero, $\psi_u = 0$ and, by Lemma 9, $Q_0 = \overline{Q}_0 = \frac{s-\mu_{h,0}}{1-\mu_{h,0})^{1+1/\sigma}}$. Plugging in into (64), one obtains

$$\operatorname{sign}\left[\Delta'_0\right] = -\frac{\gamma}{\sigma}(s-\mu_{h,0}) < 0. \tag{65}$$

Since $\psi_u = 0$ for u close to zero, the results follows by continuity. Next, we show:

R 5. *Suppose* $\Delta'_{u_0} = 0$ *for some* $u_0 \in (0, T_f]$*. Then,* Δ_u *is strictly decreasing at* u_0*.*

For this we first manipulate (64) as follows:

$$
\begin{aligned}
&\operatorname{sign}\left[\Delta_u'\right] \\
=&\operatorname{sign}\Bigg[-\frac{\gamma}{\sigma}\frac{\Delta_u}{(1-\mu_{h,u})^{1/\sigma}}\left(\mathbb{I}_{\{\psi_u=0\}}(1-\mu_{h,0})^{1+1/\sigma}+\int_{\psi_u}^{u}\rho e^{\rho t}(1-\mu_{h,t})^{1+1/\sigma}\,dt\right)+ \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \rho e^{\rho u}\left(s-\mu_{h,u}-(1-\mu_{h,u})\Delta_u\right)\Bigg] \\
=&\operatorname{sign}\Bigg[-\frac{\gamma}{\sigma}\Delta_u\left(\mathbb{I}_{\{\psi_u=0\}}e^{-\rho u}\left(\frac{1-\mu_{h,0}}{1-\mu_{h,u}}\right)^{1+1/\sigma}+\int_{\psi_u}^{u}\rho e^{-\rho(u-t)}\left(\frac{1-\mu_{h,t}}{1-\mu_{h,u}}\right)^{1+1/\sigma}dt\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\rho\left(\frac{s-\mu_{h,u}}{1-\mu_{h,u}}-\Delta_u\right)\Bigg] \\
=&\operatorname{sign}\left[-\frac{\gamma}{\sigma}\Delta_u\left(\mathbb{I}_{\{\psi_u=0\}}e^{\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]u}+\int_{\psi_u}^{u}\rho e^{\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right](u-t)}\,dt\right)+\rho\left(1-(1-s)e^{\gamma u}-\Delta_u\right)\right] \\
=&\operatorname{sign}\left[-\frac{\gamma}{\sigma}\Delta_u\left(\mathbb{I}_{\{\psi_u=0\}}e^{\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]u}+\int_{0}^{u-\psi_u}\rho e^{\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]t}\,dt\right)+\rho\left(1-(1-s)e^{\gamma u}-\Delta_u\right)\right]
\end{aligned}
$$

and where we obtain the first equality after substituting in the expression for Q_u; the second equality after dividing by $(1-\mu_{h,u})e^{\rho u}$; the third equality by using the functional form $1-\mu_{h,t}=(1-\mu_{h,0})e^{-\gamma t}$; and the fourth equality by changing variable $(x=u-t)$ in the integral. Now suppose $\Delta_u'=0$ at some u_0. From the above we have:

$$
H_{u_0}\equiv-\frac{\gamma}{\sigma}\Delta_{u_0}\left(\mathbb{I}_{\{\psi_{u_0}=0\}}e^{\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]u_0}+\int_{0}^{u_0-\psi_{u_0}}\rho e^{\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]t}\,dt\right)+\rho\left(1-(1-s)e^{\gamma u_0}-\Delta_{u_0}\right)=0.
$$

If $(1-\mu_{h,0})^{1/\sigma}Q_{u_0}<1$ then $\psi_{u_0}=0$ and $\psi'_{u_0}=0$. Together with the fact that $\Delta'_{u_0}=0$, this implies that

$$
H'_{u_0}=-\frac{\gamma}{\sigma}\Delta_{u_0}\gamma\left(1+\frac{1}{\sigma}\right)e^{\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]u_0}-\rho(1-s)\gamma e^{\gamma u_0}<0.
$$

If $(1-\mu_{h,0})^{1/\sigma}Q_{u_0}=1$, then $\psi_{u_0}=0$ and the left-derivative $\psi'_{u_0^-}=0$, so the same calculation implies that $H'_{u_0^-}<0$. If $(1-\mu_{h,0})^{1/\sigma}Q_{u_0}>1$ we first note that, around u_0,

$$
Q_u=\left(1-\mu_{h,\psi_u}\right)^{-1/\sigma}\Rightarrow\Delta_u=\left(\frac{1-\mu_{h,\psi_u}}{1-\mu_{h,u}}\right)^{1/\sigma}=e^{-\gamma\frac{\psi_u-u}{\sigma}}.
$$

So if $\Delta'_{u_0}=0$, we must have that $\psi'_{u_0}=1$. Plugging this back into H'_{u_0} we obtain that $H'_{u_0}=-\rho(1-s)\gamma e^{\gamma u_0}<0$. Lastly, if $(1-\mu_{h,0})^{1/\sigma}Q_{u_0}=1$, then the same calculation leads to $\psi_{u_0^+}=1$ and so $H_{u_0^+}<0$. In all cases, we find that H_{u_0} has strictly negative left- and right-derivatives when $H_{u_0}=0$. Thus, whenever it is equal to zero, Δ_u' is strictly decreasing. With Result R5 in mind, we then obtain:

R 6. *Δ_u' cannot change sign over $(0,T_f]$.*

Suppose it did and let u_0 be the first time in $(0, T_f]$ where Δ'_u changes sign. Because Δ'_u is continuous, we have $\Delta'_{u_0} = 0$. But recall that $\Delta'_u < 0$ for $u \simeq 0$, implying that at $u = u_0$, Δ'_u crosses the x-axis from below and is therefore increasing, contradicting Result R5. □

B.1.10 Proof of Lemma 16

With known preferences:

$$J^\star(s) = \int_0^{+\infty} \mathbb{I}_{\{u<T_s\}} e^{-ru} \left(1 - \frac{1-s}{1-\mu_{h,0}} e^{\gamma u}\right)^\sigma du.$$

Since, by definition $e^{\gamma T_s} \frac{1-s}{1-\mu_{h,0}} = 1$, we have that $T_s \to \infty$ when s goes to 1, and the integrand of $J^\star(s)$ converges pointwise towards e^{-ru}. Moreover, the integrand is bounded by e^{-ru}. Therefore, by an application of the Dominated Convergence Theorem, $J^\star(s)$ goes to $\int_0^{+\infty} e^{-ru}\, du = 1/r$ when $s \to 1$.

With preference uncertainty, for $u > 0$, we note that $Q_u(s)$ is an increasing function of s and is bounded above by $(1-\mu_{h,u})^{-1/\sigma}$. Letting $s \to 1$ in the market clearing condition (40) then shows that $Q_u \to (1-\mu_{h,u})^{-1/\sigma} > 1$. Using that $T_f > T_s$ goes to $+\infty$ when $s \to 1$, we obtain that the integrand of $J(s)$ goes to e^{-ru}. Moreover, the integrand is bounded by e^{-ru}. Therefore, by dominated convergence, $J(s)$ goes to $1/r$.

B.1.11 Proof of Lemma 17

In the market with continuous updating, we can compute:

$$J^{\star\prime}(s) = \int_0^{T_s} e^{-ru} \frac{\sigma e^{\gamma u}}{1-\mu_{h,0}} \left(1 - \frac{1-s}{1-\mu_{h,0}} e^{\gamma u}\right)^{\sigma-1} du + \frac{\partial T_s}{\partial s} \left(1 - \frac{1-s}{1-\mu_{h,0}} e^{\gamma T_s}\right)^\sigma. \tag{66}$$

The second term is equal to 0 since $e^{\gamma T_s} \frac{1-s}{1-\mu_{h,0}} = 1$. After making the change of variable $z = T_s - u$, keeping in mind that $e^{\gamma T_s} \frac{1-s}{1-\mu_{h,0}} = 1$, we obtain:

$$J^{\star\prime}(s) = \int_0^{T_s} e^{(r-\gamma)(z-T_s)} \frac{\sigma}{1-\mu_{h,0}} \left(1 - e^{-\gamma z}\right)^{\sigma-1} dz. \tag{67}$$

We then compute an approximation of $J^{\star\prime}(s)$ when $s \to 1$.

<u>When $r > \gamma$.</u> In this case we write:

$$J^{\star\prime}(s) = \int_0^{T_s/2} e^{(r-\gamma)(z-T_s)} \frac{\sigma}{1-\mu_{h,0}} \left(1 - e^{-\gamma z}\right)^{\sigma-1} dz + \int_{T_s/2}^{T_s} e^{(r-\gamma)(z-T_s)} \frac{\sigma}{1-\mu_{h,0}} \left(1 - e^{-\gamma z}\right)^{\sigma-1} dz.$$

The first term is less than $e^{-(r-\gamma)T_s/2} \frac{\sigma}{1-\mu_{h,0}} \int_0^{T_s/2} [1 - e^{-\gamma z}]\, dz$. The integrand goes to 1 as z goes to infinity and so, by Cesàro summation, the integral is equivalent to $T_s/2$, which is dominated by $e^{-(r-\gamma)T_s/2}$ as $s \to 1$ and $T_s \to \infty$. Thus, the first term converges to zero as $s \to 1$. The second term can be written:

$$\sigma \int_0^{\infty} \mathbb{I}_{\{u \leq T_s/2\}} e^{-(r-\gamma)u} \left(1 - \frac{1-s}{1-\mu_{h,0}} e^{\gamma u}\right)^{\sigma-1} du.$$

Since T_s goes to infinity when s goes to 1, the integrand goes to , and is bounded by, $e^{-(r-\gamma)u}(1 - \sqrt{\frac{1-s}{1-\mu_{h,0}}})^{\sigma-1}$. Therefore, by dominated convergence, $J^{\star\prime}(s)$ goes to $\frac{\sigma}{(1-\mu_{h,0})(r-\gamma)}$.

When $r = \gamma$. Then we have:

$$J^{\star\prime}(s) = \sigma \int_0^{T_s} \left(1 - e^{-\gamma z}\right)^{\sigma-1} dz.$$

The integrand goes to 1 when T_s goes to infinity. Thus, the Cesàro mean $I'(s)/T_s$ converges to σ, i.e.:

$$J^{\star\prime}(s) \sim \sigma T_s = -\frac{\sigma}{\gamma} \log\left(\frac{1-s}{1-\mu_{h,0}}\right).$$

When $r < \gamma$. In that case:

$$J^{\star\prime}(s) = \sigma e^{(\gamma-r)T_s} \int_0^{+\infty} \mathbb{I}_{\{z<T_s\}} e^{-(\gamma-r)z} \left(1 - e^{-\gamma z}\right)^{\sigma-1} dz,$$

The integrand in the second line goes to, and is bounded by, $e^{-(\gamma-r)z}(1-e^{-\gamma z})^{\sigma-1}$, which in integrable. Therefore, by dominated convergence, the integral goes to $\int_0^{+\infty} e^{-(\gamma-r)z} \left(1 - e^{-\gamma z}\right)^{\sigma-1} dz$ when s goes to 1. Finally, using that $e^{-\gamma T_s} = \frac{1-s}{1-\mu_{h,0}}$, we obtain:

$$J^{\star\prime}(s) \sim \sigma \left(\frac{1-\mu_{h,0}}{1-s}\right)^{1-r/\gamma} \int_0^{+\infty} e^{-(\gamma-r)z} \left(1 - e^{-\gamma z}\right)^{\sigma-1} dz.$$

B.1.12 Proof of Lemma 18

Throughout all the proof and the intermediate results therein, we work under the maintained assumption

$$\gamma + \gamma/\sigma - \rho > 0 \iff \gamma + \sigma(\gamma - \rho) > 0, \tag{68}$$

which is without loss of generality since we want to compare prices when σ is close to zero. We start by differentiating $J(s)$:

$$J'(s) = \frac{\partial T_f}{\partial s} e^{-rT_f} e^{-\gamma T_f} Q^{\sigma}_{T_f^-} + \int_0^{T_f} e^{-ru} e^{-\gamma u} \frac{\partial Q^{\sigma}_u}{\partial s} du > \int_{T_1}^{T_2} e^{-ru} e^{-\gamma u} \frac{\partial Q^{\sigma}_u}{\partial s} du,$$

where the inequality follows from the following facts: the first term is zero since $Q_{T_f^-} = 0$; the integrand in the second term is positive since Q_u is increasing in s by equation (40); and $0 < T_1 < T_2 < T_f$ are defined as in the paragraph following Lemma 11, as follows. We consider that s is close to 1 so that $Q_u > 1$ for some u. Then, $T_1 < T_2$ are defined as the two solutions of $Q_{T_1} = Q_{T_2} = 1$. Note that T_1 and T_2 are also the two solutions of $\overline{Q}_{T_1} = \overline{Q}_{T_2}$. Because both Q_u and $\overline{Q}_u$ are hump shaped, we know that Q_u and $\overline{Q}_u$ are strictly greater than one for $u \in (T_1, T_2)$, and less than one otherwise. For $u \in (T_1, T_2)$, we can define $\psi_u > 0$ as in the paragraph following Lemma 11: $Q_u = (1 - \mu_{h\psi_u})^{-1/\sigma}$. By construction, $\psi_u \in (0, u)$, and, as shown in Section B.1.13:

$$\frac{\partial \psi_u}{\partial s} = \frac{\gamma + \sigma(\gamma - \rho)}{\gamma\rho} \frac{(1 - e^{-\rho u})\, e^{\gamma u}}{e^{-(\rho-\gamma)(u-\psi_u)} - e^{-(\gamma/\sigma)(u-\psi_u)}}. \tag{69}$$

Plugging $Q_u^\sigma = (1 - \mu_{h\psi_u})^{-1} = e^{\gamma\psi_u}$ in the expression of $J'(s)$, we obtain:

$$J'(s) > \frac{\gamma + \sigma(\gamma - \rho)}{\rho} \int_{T_1}^{T_2} e^{-ru} \frac{(1 - e^{-\rho u})\, e^{\gamma\psi_u}}{e^{-(\rho-\gamma)(u-\psi_u)} - e^{-(\gamma/\sigma)(u-\psi_u)}}\, du. \tag{70}$$

When $r > \gamma$. For this case fix some $\overline{u} > 0$ and pick s close enough to one so that that $Q_{\overline{u}} > 1$. Such s exists since, as argued earlier in Section B.1.10, for all $u > 0$, $Q_u \to (1 - \mu_{h,u})^{-1/\sigma}$ as $s \to 1$. Since the integrand in (70) is strictly positive, we have:

$$\begin{aligned}
J'(s) &> \frac{\gamma + \sigma(\gamma - \rho)}{\rho} \int_0^{\overline{u}} \mathbb{I}_{\{u>T_1\}} e^{-ru} \frac{(1 - e^{-\rho u})\, e^{\gamma\psi_u}}{e^{-(\rho-\gamma)(u-\psi_u)} - e^{-(\gamma/\sigma)(u-\psi_u)}}\, du \\
&> \frac{\gamma + \sigma(\gamma - \rho)}{\rho} \frac{1}{e^{|\rho-\gamma|(\overline{u}-\psi_{\overline{u}})} - e^{-(\gamma/\sigma)(\overline{u}-\psi_{\overline{u}})}} \int_0^{\overline{u}} \mathbb{I}_{\{u>T_1\}} e^{-ru} \left(1 - e^{-\rho u}\right) e^{\gamma\psi_u}\, du.
\end{aligned}$$

where the second line follows from the fact, proven is Section B.1.13, that $u - \psi_u$ is strictly increasing in u when $\psi_u > 0$. In Section B.1.13 we also prove that $T_1 \to 0$ and that, for all $u > 0$, $\psi_u \to u$ when s goes to 1. Therefore, in the above equation, the integral remains bounded away from zero, and the whole expression goes to infinity.

When $r \leq \gamma$. In this case we make the change of variable $z \equiv T_s - u$ in equation (70) and we use that $e^{-\gamma T_s} = \frac{1-s}{1-\mu_{h,0}}$:

$$\begin{aligned}
J'(s) &> \frac{\gamma + \sigma(\gamma - \rho)}{\rho} \int_{T_s-T_2}^{T_s-T_1} \left(\frac{1-s}{1-\mu_{h,0}}\right)^{\frac{r}{\gamma}} e^{rz} \frac{\left(1 - e^{-\rho(T_s-z)}\right) e^{\gamma\psi_{T_s-z}}}{e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}}\, dz \\
&> \frac{\gamma + \sigma(\gamma - \rho)}{\rho} \int_0^{+\infty} \mathbb{I}_{\{\max\{T_s-T_2,0\}<z<T_s-T_1\}} \left(\frac{1-s}{1-\mu_{h,0}}\right)^{\frac{r}{\gamma}} \\
&\qquad\qquad e^{rz} \frac{\left(1 - e^{-\rho(T_s-z)}\right) e^{\gamma\psi_{T_s-z}}}{e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}}\, dz.
\end{aligned}$$

where the second line follows from the addition of the max operator in the indicator variable and the fact that the integrand is strictly positive. We show in Section B.1.13 that, if $\psi_{T_s-z} > 0$, then:

$$e^{\gamma\psi_{T_s-z}} > \begin{cases} \left(\frac{\gamma+\sigma(\gamma-\rho)}{\rho}\right)^{\frac{\gamma}{\rho-\gamma}} \left(\frac{1-s}{1-\mu_{h,0}}\right)^{-1} e^{-\gamma z} & \text{if } \rho \neq \gamma, \\ e^{-(1+\sigma)} \left(\frac{1-s}{1-\mu_{h,0}}\right)^{-1} e^{-\gamma z} & \text{if } \rho = \gamma, \end{cases} \tag{71}$$

and:

$$\left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}\right)^{-1} > \frac{\min\{\gamma,\rho\}}{\gamma+\sigma(\gamma-\rho)}. \tag{72}$$

When $\gamma \neq \rho$, we obtain:

$$\begin{aligned} J'(s) > \left(\frac{\gamma+\sigma(\gamma-\rho)}{\rho}\right)^{\frac{\gamma}{\rho-\gamma}} \min\{\gamma/\rho, 1\} \left(\frac{1-s}{1-\mu_{h,0}}\right)^{-1+\frac{r}{\gamma}} \\ \times \int_0^{+\infty} \mathbb{I}_{\{\max\{T_s-T_2,0\}<z<T_s-T_1\}} e^{-(\gamma-r)z} \left(1-e^{-\rho(T_s-z)}\right) dz. \end{aligned} \tag{73}$$

Consider first the case $\gamma < r$. In Section B.1.13 we show that $T_s - T_2 < 0$ when s is close to 1 and that T_1 goes to 0 when s goes to 1. Since T_s goes to infinity, these facts imply that the integrand goes to, and is bounded above by, $e^{-(\gamma-r)z}$ when $s \to 1$. Therefore, by dominated convergence, the integral goes to $1/(\gamma - r)$. A similar computation obtains when $\gamma = \rho$.

Consider now the case $\gamma = r$. When $\gamma \neq \rho$, equation (73) rewrites:

$$\begin{aligned} J'(s) &> \left(\frac{\gamma+\sigma(\gamma-\rho)}{\rho}\right)^{\frac{\gamma}{\rho-\gamma}} \min\{\gamma/\rho, 1\} \int_{\max\{T_s-T_2,0\}}^{T_s-T_1} \left(1-e^{-\rho(T_s-z)}\right) dz \\ &= \left(\frac{\gamma+\sigma(\gamma-\rho)}{\rho}\right)^{\frac{\gamma}{\rho-\gamma}} \min\{\gamma/\rho, 1\} \left(T_s - T_1 - \max\{T_s-T_2, 0\} - \frac{e^{-\rho T_1} - e^{-\rho\min\{T_2,T_s\}}}{\rho}\right). \end{aligned}$$

Since $T_s - T_2 < 0$ and $T_1 \to 0$ when s goes to 1, the last term in large parenthesis is equivalent to $T_s = \log((1-s)^{-1})/\gamma$ when s goes to 1. A similar computation obtains when $\gamma = \rho$.

B.1.13 Intermediate results for the proofs of Lemma 16, 17 and 18

Derivative of the ψ_u function when $\psi_u > 0$. When $\psi_u > 0$, time–τ_u low–valuation investors hold $q_{\tau_u,u} = 1$ if $\tau_u < \psi_u$, and $q_{\tau_u,u} = (1-\mu_{h,\tau_u})^{1/\sigma}(1-\mu_{h\psi_u})^{-1/\sigma}$ if $\tau_u > \psi_u$. The market clearing condition (40) rewrites:

$$\begin{aligned} &1-\mu_{h,0} + \int_0^{\psi_u} \rho e^{\rho t}(1-\mu_{h,t})\, dt + \int_{\psi_u}^{u} \rho e^{\rho t}(1-(1-\mu_{h,0})\mu_{h,t})^{1+1/\sigma}(1-\mu_{h\psi_u})^{-1/\sigma}\, dt \\ =&s-\mu_{h,0} + \int_0^u \rho e^{\rho t}(s-\mu_{h,t})\, dt. \end{aligned} \tag{74}$$

We differentiate this equation with respect to s:

$$\frac{\partial \psi_u}{\partial s}\frac{\gamma}{\sigma}\int_{\psi_u}^{u} e^{\rho t}(1-\mu_{h,t})^{1+1/\sigma}(1-\mu_{h\psi_u})^{-1/\sigma}\,dt = \int_0^u e^{\rho t}\,dt.$$

After computing the integrals and rearranging the terms we obtain equation (69).

Limits of T_1 and T_2 when $s \to 1$. For any $u > 0$, when s is close enough to 1 we have $Q_u > 1$ and thus $T_1 < u < T_2$. Therefore $T_1 \to 0$ and $T_2 \to \infty$, when $s \to 1$. To obtain that $T_2 > T_s$ when s is close to 1, it suffices to show that $\overline{Q}_{T_s} > 1$ for s close to 1. After computing the integrals in equation (41) and using that $e^{-\gamma T_s} = \frac{1-s}{1-\mu_{h,0}}$, we obtain:

$$\overline{Q}_{T_s} = \frac{N_s}{D_s},$$

where

$$N_s = \begin{cases} (1-s)\frac{\gamma}{\rho-\gamma} + \frac{\gamma}{\gamma-\rho}(1-\mu_{h,0})\left(\frac{1-s}{1-\mu_{h,0}}\right)^{\rho/\gamma} & \text{if } \rho \neq \gamma \\ (1-s)\log\left(\frac{1-\mu_{h,0}}{1-s}\right) & \text{if } \rho = \gamma \end{cases}$$

$$D_s = \frac{\sigma(1-\mu_{h,0})^{1+1/\sigma}}{\gamma+\sigma(\gamma-\rho)}\left\{\gamma\left(1+\frac{1}{\sigma}\right)\left(\frac{1-s}{1-\mu_{h,0}}\right)^{\rho/\gamma} - \rho\left(\frac{1-s}{1-\mu_{h,0}}\right)^{1+1/\sigma}\right\}.$$

When $\gamma \leq \rho$, $\overline{Q}_{T_s}$ goes to infinity when s goes to 1. When $\gamma > \rho$, $\overline{Q}_{T_s}$ goes to $\frac{\gamma+\sigma(\gamma-\rho)}{\sigma(\gamma-\rho)(1-\mu_{h,0})^{1+1/\sigma}} > 1$.

Proof that $u - \psi_u$ is strictly increasing in u when $\psi_u > 0$. Rearranging (74), we obtain:

$$\frac{1-s}{1-\mu_{h,0}}e^{\rho u} = \int_{\psi_u}^{u} \rho e^{(\rho-\gamma)t}\,dt - e^{\frac{\gamma}{\sigma}\psi_u}\int_{\psi_u}^{u} \rho e^{\left[\rho-\gamma\left(1+\frac{1}{\sigma}\right)\right]t}\,dt.$$

When $\rho \neq \gamma$, calculating the integrals and reorganizing terms leads to

$$\frac{1-s}{1-\mu_{h,0}}\frac{e^{\gamma u}}{\rho} = \left(\frac{1}{\rho-\gamma} + \frac{1}{\gamma\left(1+\frac{1}{\sigma}\right)-\rho}\right)\left(1-e^{-(\rho-\gamma)(u-\psi_u)}\right) - \frac{1}{\gamma\left(1+\frac{1}{\sigma}\right)-\rho}\left(1-e^{-\frac{\gamma}{\sigma}(u-\psi_u)}\right) \quad (75)$$

Taking the derivative of the right-hand side with respect to $u - \psi_u$ we easily obtain that it is strictly increasing in $u - \psi_u$, given our parameter restriction that $\gamma > \sigma(\gamma - \rho)$. Since the right-hand side is strictly increasing in u, then $u - \psi_u$ is a strictly increasing function of u. When $\rho = \gamma$, the left-hand side stays the same and the right-hand side becomes

$$u - \psi_u + \frac{\sigma}{\gamma}\left(e^{-\frac{\gamma}{\sigma}(u-\psi_u)} - 1\right)$$

which is strictly increasing in $u - \psi_u$ as well, implying that $u - \psi_u$ is a strictly increasing function of u.

Proof that $\psi_u \to u$ when $s \to 1$. As noted earlier in Section B.1.10, for any u, $Q_u \to (1-\mu_{h,u})^{-1/\sigma}$ as $s \to 1$. Together with the defining equation of ψ_u, $Q_u = (1-\mu_{h\psi_u})^{1/\sigma}$, this implies that $\psi_u \to u$ as $s \to 1$.

Proof of equation (71). When $\gamma \neq \rho$, we make the change of variable $z \equiv T_s - u$ in the market clearing condition (75):

$$\frac{e^{-\gamma z}}{\rho} = \frac{1}{\rho-\gamma} - \left(\frac{1}{\rho-\gamma} + \frac{1}{\gamma\left(1+\frac{1}{\sigma}\right)-\rho}\right) e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} + \frac{1}{\gamma\left(1+\frac{1}{\sigma}\right)-\rho} e^{-\frac{\gamma}{\sigma}\left(T_s-z-\psi_{T_s-z}\right)}, \quad (76)$$

where we have used that $e^{-\gamma T_s} = \frac{1-s}{1-\mu_{h,0}}$. This implies that:

$$\begin{aligned}
\frac{e^{-\gamma z}}{\rho} &> \frac{1}{\rho-\gamma} - \left(\frac{1}{\rho-\gamma} + \frac{1}{\gamma\left(1+\frac{1}{\sigma}\right)-\rho}\right) e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} \\
&= \frac{1}{\rho-\gamma} - \frac{\frac{\gamma}{\sigma}}{(\rho-\gamma)\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]} e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})}
\end{aligned}$$

Using $e^{-\gamma T_s} = \frac{1-s}{1-\mu_{h,0}}$ and doing some algebra, we arrive at:

$$\frac{\rho}{(\rho-\gamma)\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]} e^{(\rho-\gamma)\psi_{T_s-z}} > \left(\frac{1-s}{1-\mu_{h,0}}\right)^{-\frac{\rho-\gamma}{\gamma}} \frac{e^{-(\rho-\gamma)z}}{\rho-\gamma}.$$

Equation (71) for $\gamma \neq \rho$ follows. Finally, when $\gamma = \rho$, the same manipulations lead to:

$$e^{-\gamma z} = \gamma\left(T_s - z - \psi_{T_s-z}\right) - \sigma + \sigma e^{-\frac{\gamma}{\sigma}(T_s-z-\psi_{T_s-z})} \Rightarrow 1 > e^{-\gamma z} > \gamma(T_s - z - \psi_{T_s-z}) - \sigma.$$

Taking exponentials on both sides, and using $e^{-\gamma T_s} = \frac{1-s}{1-\mu_{h,0}}$, lead to equation (71) for $\gamma = \rho$.

Proof of equation (72). When $\gamma \neq \rho$, we write equation (76) as follows:

$$\frac{1}{\rho-\gamma} - \frac{e^{-\gamma z}}{\rho} = \left(\frac{1}{\rho-\gamma} + \frac{1}{\gamma\left(1+\frac{1}{\sigma}\right)-\rho}\right) e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - \frac{1}{\gamma\left(1+\frac{1}{\sigma}\right)-\rho} e^{-\frac{\gamma}{\sigma}\left(T_s-z-\psi_{T_s-z}\right)}$$

When $\rho > \gamma$, we add $-\frac{1}{\rho-\gamma} \times e^{-(\gamma/\sigma)(T_s-z-\psi_{T_s-z})}$, which is negative, to the right–hand side:

$$\begin{aligned}
& \frac{1}{\rho-\gamma} - \frac{e^{-\gamma z}}{\rho} > \frac{\frac{\gamma}{\sigma}}{(\rho-\gamma)\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]} \left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-\frac{\gamma}{\sigma}\left(T_s-z-\psi_{T_s-z}\right)}\right) \\
\Longrightarrow \quad & \frac{1}{\rho-\gamma} > \frac{\frac{\gamma}{\sigma}}{(\rho-\gamma)\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]} \left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-\frac{\gamma}{\sigma}\left(T_s-z-\psi_{T_s-z}\right)}\right) \\
\Longrightarrow \quad & \left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-\frac{\gamma}{\sigma}\left(T_s-z-\psi_{T_s-z}\right)}\right)^{-1} > \frac{\gamma}{\gamma+\sigma(\gamma-\rho)},
\end{aligned}$$

where we can keep the inequality the same because $\rho > \gamma$. Equation (72) when $\rho > \gamma$ follows.

When $\rho < \gamma$, we can also add $-\frac{1}{\rho-\gamma} \times e^{-(\gamma/\sigma)(T_s - z - \psi_{T_s - z})}$ to the right hand side. But since this term is now negative, we obtain:

$$\begin{aligned}
& \frac{1}{\rho-\gamma} - \frac{e^{-\gamma z}}{\rho} < \frac{\frac{\gamma}{\sigma}}{(\rho-\gamma)\left[\gamma\left(1+\frac{1}{\sigma}\right)-\rho\right]}\left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-\frac{\gamma}{\sigma}\left(T_s-z-\psi_{T_s-z}\right)}\right) \\
\implies \quad & 1 - e^{-\gamma z}\frac{\rho-\gamma}{\rho} > \frac{\frac{\gamma}{\sigma}}{\gamma\left(1+\frac{1}{\sigma}\right)-\rho}\left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-\frac{\gamma}{\sigma}\left(T_s-z-\psi_{T_s-z}\right)}\right) \\
\implies \quad & \frac{\gamma}{\rho} > \frac{\frac{\gamma}{\sigma}}{\gamma\left(1+\frac{1}{\sigma}\right)-\rho}\left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-\frac{\gamma}{\sigma}\left(T_s-z-\psi_{T_s-z}\right)}\right) \\
\implies \quad & \left(e^{-(\rho-\gamma)(T_s-z-\psi_{T_s-z})} - e^{-\frac{\gamma}{\sigma}\left(T_s-z-\psi_{T_s-z}\right)}\right)^{-1} > \frac{\rho}{\gamma+\sigma(\gamma-\rho)}.
\end{aligned}$$

where we use $e^{-\gamma z} < 1$ to move from the second to the third line. Equation (72), when $\rho > \gamma$, follows. Finally, when $\gamma = \rho$, equation (72) follows since $1 - e^{-(\gamma/\sigma)(T_s - z - \psi_{T_s - z})} < 1$.

B.2 Trading profits

Consider, in the analytical example, a trader who learns at some time T that she has a high valuation. Assume for simplicity that $T < T_f$ so that the investor find it optimal to hold 1 unit at this time. The trading profits can be defined as:

$$\Pi = -\int_0^T p_t dq_t.$$

After integrating by part we obtain:

$$\begin{aligned}\Pi &= -p_T q_T + p_0 s + \int_0^T \dot{p}_t q_t \, dt = -p_t + p_0 s + \int_0^T \dot{p}_t q_t \, dt \\ &= -p_0 + \int_0^T \dot{p}_t \, dt + p_0 s + \int_0^T \dot{p}_t q_t \, dt = -p_0(1-s) + \int_0^T \dot{p}_t \left[q_t - 1\right] \, dt.\end{aligned}$$

Now in term of holding plans this can be written:

$$\Pi = -p_0(1-s) + \int_0^T \dot{p}_u \left[q_{\ell,\tau_u,u} - 1\right] \, du < 0.$$

Note that trading profits are negative. This makes sense because, in this model, every trader who ends up purchasing before T_f is a net buyer: she starts with s and ends with 1. This is in contrast with models of liquidity provision, in which trading profits are positive.

Note also that, since traders are net buyers, the best way to minimize cost would be to buy immediately $1-s$ at time zero. Of course, although this maximizes trading profits, this strategy does not maximize inter temporal utility, because it requires the trader to incur large holding costs during the liquidity shock.

Next, let us calculate the expectations of Π conditional on the event that there are exactly n updates over $[0,T)$. For this we need to figure out the distribution of τ_u conditional on n updates over $[0,T)$. Note first that:

$$\begin{aligned}\text{Proba}(\tau_u \leq t \wedge N_T = n) &= \sum_{k=0}^{n} \text{Proba}(N_t = k \wedge N_u - N_t = 0 \wedge N_T - N_u = n-k) \\ &= \sum_{k=0}^{n} \frac{e^{-\rho t}(\rho t)^k}{k!} e^{-\rho(u-t)} \frac{e^{-\rho(T-u)}(\rho(T-u))^{n-k}}{(n-k)!} \\ &= \frac{e^{-\rho T}(\rho T)^n}{n!} \sum_{k=0}^{n} C_n^k \left(\frac{t}{T}\right)^k \left(\frac{T-u}{T}\right)^{n-k} \\ &= \text{Proba}(N_t = n) \left[1 - \frac{u-t}{T}\right]^n.\end{aligned}$$

Therefore the distribution of τ_u conditional on n updates over $[0, T)$ is

$$Pr(\tau_u \leq t \,|\, N_T = n) = \left[1 - \frac{u-t}{T}\right]^n.$$

One sees that an increase in n creates a first-order stochastic dominance shift in the distribution. This is intuitive: if there has been lots of updates, then it is more likely that the last update before u is close to u. Combined with the observation that $q_{\ell,t,u}$ is decreasing in t, this implies that the expectations of Π conditional on n updates before T is decreasing. This implies that $\mathbb{E}\left[\Pi \,|\, n, T \leq T_f\right]$ is decreasing in n.

B.3 Private information about common values reduces trading volume

In general, private information about common values reduces trading volume, because it generates adverse selection. Below, we illustrate this point in a noisy rational expectations model, adapted from Grossman and Stiglitz (1980). The main difference is that, while in Grossman and Stiglitz (1980) there are noise traders, in the present case all investors are rational. Trading occurs, in equilibrium, because of endowment shocks generating potential gains from trade. This is an important difference for the analysis of trading volume with private information. Since noise traders do not optimize, they don't respond to increased adverse selection.

The model. Let us consider a simple version of Grossman and Stiglitz (1980). There is one asset with random payoff $v \sim \mathcal{N}(0, 1/\Psi_v)$. There are λ informed investors and $1-\lambda$ uninformed ones, all with Constant Absolute Risk Aversion (CARA) utility, α. Uninformed investors receive no signal and no endowment. Informed investors observe signal

$$v + \frac{\varepsilon}{\sqrt{\Psi_\varepsilon}}, \tag{77}$$

and have random endowment s/λ, where $s \sim \mathcal{N}(0, 1/\Psi_s)$. As is standard, the common but random component of the endowment shock prevents uninformed investor from perfectly inferring informed investors' information from the asset price. The factor $1/\lambda$ keeps the aggregate supply equal to s as we vary the fraction of informed investors.

Equilibrium. To solve the model, we guess and verify that, to an uninformed investor, the price is observationally equivalent to a signal of the form:

$$v + \frac{\varepsilon}{\sqrt{\Psi_\varepsilon}} - \frac{s}{\theta}, \tag{78}$$

for some $\theta > 0$ to be determined in equilibrium. Note in particular that the coefficient on s is negative: when they receive a larger endowment, the informed investors want to sell more. This puts downward pressure on the price. But uninformed investors do not know whether the downward pressure originates from an endowment shock or from adverse information about v. Thus, they will rationally interpret this negative price pressure as a noisy signal that the fundamental value has gone down.

Straightforward calculations show that the precision of the price signal, (78) is

$$\Psi_p = \Psi_\varepsilon \frac{\Psi_s \theta^2}{\Psi_s \theta^2 + \Psi_\varepsilon} < \Psi_\varepsilon.$$

Clearly, because of the noisy supply, the precision of the price signal, (78), is lower than that of informed investors' signal, (77). The demand of informed and uninformed investors can be written:

$$D_I = \frac{\mathbb{E}_I[v] - p}{\alpha \mathbb{V}_I[v]} - \frac{s}{\lambda}, \quad \text{and} \quad D_U = \frac{\mathbb{E}_U[v] - p}{\alpha \mathbb{V}_U[v]}.$$

Using Bayes' rule, and keeping in mind that the prior has mean zero, we obtain that the posterior mean of informed and uninformed investors are

$$\mathbb{E}_I[v] = \frac{\Psi_\varepsilon}{\Psi_\varepsilon + \Psi_v}\left[v + \frac{\varepsilon}{\sqrt{\Psi_\varepsilon}}\right], \quad \text{and} \quad \mathbb{E}_U[v] = \frac{\Psi_p}{\Psi_p + \Psi_v}\left[v + \frac{\varepsilon}{\sqrt{\Psi_\varepsilon}} - \frac{s}{\theta}\right].$$

The posterior variances of informed and uninformed investors are

$$\mathbb{V}_I[v] = (\Psi_v + \Psi_\varepsilon)^{-1}, \quad \text{and} \quad \mathbb{V}_U[v] = (\Psi_v + \Psi_p)^{-1}.$$

Therefore, the demand of informed and uninformed investors can be written:

$$\begin{aligned} D_I &= \frac{1}{\alpha}\left[\Psi_\varepsilon\left(v + \frac{\varepsilon}{\sqrt{\Psi_\varepsilon}}\right) - (\Psi_\varepsilon + \Psi_v)\, p\right] - \frac{s}{\lambda} \\ D_U &= \frac{1}{\alpha}\left[\Psi_p\left(v + \frac{\varepsilon}{\sqrt{\Psi_\varepsilon}} - \frac{s}{\theta}\right) - (\Psi_p + \Psi_v)\, p\right]. \end{aligned}$$

Solving for the price in $\lambda D_I + (1-\lambda) D_U = 0$, we obtain:

$$\begin{aligned} p = &- \frac{\alpha s}{\lambda \Psi_\varepsilon + (1-\lambda)\Psi_p + \Psi_v} \\ &+ \frac{\lambda \Psi_\varepsilon}{\lambda \Psi_\varepsilon + (1-\lambda)\Psi_p + \Psi_v}\left(v + \frac{\varepsilon}{\sqrt{\Psi_\varepsilon}}\right) + \frac{(1-\lambda)\Psi_p}{\lambda \Psi_\varepsilon + (1-\lambda)\Psi_p + \Psi_v}\left(v + \frac{\varepsilon}{\sqrt{\Psi_\varepsilon}} - \frac{s}{\theta}\right). \end{aligned}$$

After a couple of lines of algebra we see that our guess is verified iff:

$$\theta = \frac{\lambda \Psi_\varepsilon}{\alpha}.$$

The Volume. The aggregate demand from uninformed investors is

$$(1-\lambda) D_U = -\frac{\lambda(1-\lambda)\Psi_v(\Psi_\varepsilon - \Psi_p)}{\lambda \Psi_\varepsilon + (1-\lambda)\Psi_p + \Psi_s}\frac{1}{\alpha}\left\{v + \frac{\varepsilon}{\sqrt{\Psi_\varepsilon}} - \frac{s}{\theta}\right\}.$$

Without asymmetric information, it would be equal to $(1-\lambda)s$: indeed, the equilibrium allocation in this case dictates that there is full risk sharing, and hence that all investors, informed and uninformed, hold s shares of the assets.[24]

We would like to know whether this trading volume increases or decreases with asymmetric infor-

[24]Note that with symmetric information, the equilibrium volume is the same regardless of the level of risk (as long as it is positive). Indeed, with CARA agents, in the setup considered, the equilibrium allocation prescribes that agents share risk equally, regardless of their (positive) risk aversion and regardless of the level of risk.

mation. One sees that there are competing effects. On the one hand, the loading of the order flow, D_U, on s, is equal to

$$(1-\lambda)\left[1-\frac{\Psi_p}{\Psi_\varepsilon}\right]\frac{\Psi_v}{\lambda\Psi_\varepsilon+(1-\lambda)\Psi_p+\Psi_v}<1-\lambda.$$

That is, asymmetric information reduces the "fundamental" trading volume associated with hedging needs. For example, suppose that $v=\varepsilon=0$. Then, when s is positive, the informed investors want to sell assets, which puts downward pressure on the price. Uninformed investors rationnaly interpret the low price as a bad signal about the fundamental value of the asset, and reduce their demand relative to the full information case. In equilibrium, uninformed investors end up purchasing less asset from informed investors than they would have under symmetric infomaiton.

While there is less trading for fundamental "hedging" motives, there is now some speculative trading. For example, suppose that v is positive, but $\varepsilon=s=0$. Then both the informed and the uninformed investors receive a positive signal about the fundamental value of the asset. But the informed investor views his signal as more precise: hence, if the uninformed investor demand is positive, the informed demand will be positive as well. Thus, market clearing implies that the price must adjust so that uninformed demand must be negative, and informed demand must be positive.

Our main result is that:

Proposition 11. *The volume is smaller under asymmetric than under symmetric information:*

$$(1-\lambda)\mathbb{V}\left[D_U\right]<\frac{1-\lambda}{\Psi_s}.$$

To show this, we start from:

$$\mathbb{V}\left[D_U\right]=\left(\frac{\lambda(1-\lambda)\Psi_v(\Psi_p-\Psi_\varepsilon)}{\lambda\Psi_\varepsilon+(1-\lambda)\Psi_p+\Psi_v}\right)^2\frac{1}{\alpha^2}\left\{\frac{1}{\Psi_v}+\frac{1}{\Psi_\varepsilon}+\frac{1}{\theta^2\Psi_s}\right\}.$$

Substituting in $\alpha^2=\lambda^2\Psi_\varepsilon^2/\theta^2$:

$$\mathbb{V}\left[D_U\right]=\frac{(1-\lambda)^2}{\Psi_s}\left(\frac{\Psi_v(1-\Psi_p/\Psi_\varepsilon)}{\lambda\Psi_\varepsilon+(1-\lambda)\Psi_p+\Psi_v}\right)^2\left\{\frac{\theta^2\Psi_s}{\Psi_v}+\frac{\theta^2\Psi_s}{\Psi_\varepsilon}+1\right\}.$$

Now using the formula for Ψ_p we have that $\theta^2\Psi_s=\Psi_p/(1-\Psi_p/\Psi_\varepsilon)$. Plugging this in we have:

$$
\begin{aligned}
\mathbb{V}\left[D_U\right] &= \frac{(1-\lambda)^2}{\Psi_s}\left(\frac{\Psi_v(1-\Psi_p/\Psi_\varepsilon)}{\lambda\Psi_\varepsilon+(1-\lambda)\Psi_p+\Psi_v}\right)^2 \frac{\Psi_p(\Psi_v+\Psi_\varepsilon)+\Psi_v\Psi_\varepsilon(1-\Psi_p/\Psi_\varepsilon)}{\Psi_v\Psi_\varepsilon(1-\Psi_p/\Psi_\varepsilon)} \\
&= \frac{(1-\lambda)^2}{\Psi_s}\frac{\Psi_v(1-\Psi_p/\Psi_\varepsilon)}{(\lambda\Psi_\varepsilon+(1-\lambda)\Psi_p+\Psi_v)^2}\left(\Psi_p+\Psi_v\right) \\
&= \frac{(1-\lambda)^2}{\Psi_s}\times(1-\Psi_p/\Psi_\varepsilon)\times\frac{\Psi_v}{\lambda\Psi_\varepsilon+(1-\lambda)\Psi_p+\Psi_\varepsilon}\times\frac{\Psi_p+\Psi_v}{\Psi_p+\Psi_v+\lambda(\Psi_\varepsilon-\Psi_p)}
\end{aligned}
$$

Clearly, all terms multiplying $(1-\lambda)^2/\Psi_s$ are less than one, establishing the claim. □

B.4 Information collection effort

In this appendix we study a simple static variant of our model, with three stages: *ex-ante* banks choose how much information collection effort to exert, *interim* banks receive a signal about their preferences and trade in a centralized market, *ex-post* banks discover their types and payoffs realize. In this context, again, we find that the equilibrium is constrained Pareto efficient, i.e., both the choice of effort, and the allocation coincide with the one that a social planner would choose.

B.4.1 Setup

Consider a continuum of banks with utility $v(\theta, q)$ for holding an asset in supply s. Assume bank type can be either high or low, $\theta \in \{\theta_\ell, \theta_h\}$ and that the utility function satisfies the same regularity conditions as in the paper. There are three stages: *ex-ante* and *interim* and *ex-post*. In the first stage all banks start with endowment equal to s, and they invest in information collection efforts. In the second stage, banks receive a signal about their type and trade assets in a centralized market. In the third stage, banks discover their types and payoffs realize.

To model information collection effort, we assume that a bank can choose the probability ρ of knowing its type for sure. Namely, we assume that a bank observes its type exactly with probability ρ, i.e., it receives the signal $s = h$ if it has a high type, or $s = \ell$ if it has a low type. With the complementary probability, $1 - \rho$, the bank observes no signal, which we indicate using the shorthand $s = m$. Just as in our main dynamic model, banks who observe $s = m$ face preference uncertainty: they believe that they have a high type with probability μ, and a low type with probability $1 - \mu$.

Assume for now that all banks choose the same level of effort (we will argue later that this is without loss of generality). An allocation of asset is a vector $\{q_s\}_{s\in\{\ell,m,h\}}$, prescribing that a bank who observes signal $s \in \{\ell, m, h)$ holds a quantity q_s of assets. An allocation is feasible if

$$\rho \left[\mu q_h + (1 - \mu) q_\ell\right] + (1 - \rho) q_m = s, \tag{79}$$

where ρ is the level of effort chosen by banks.

B.4.2 Social planning problem

We define the social planning problem in two steps. First, given any level of effort, ρ, the planner solves, at the *interim* stage:

$$W(\rho) = \max_{\{q_s\}} \rho \left[\mu v(\theta_h, q_h) + (1 - \mu) v(\theta_\ell, q_\ell)\right] + (1 - \rho) \left[\mu v(\theta_h, q_m) + (1 - \mu) v(\theta_\ell, q_m)\right],$$

subject to (79). At the *ex-ante* stage, the planner solves:

$$\max_{\rho\in[0,1]} W(\rho) - C(\rho),$$

where $C(\rho)$ is a continuously differentiable and strictly convex function of ρ. Clearly, since $W(\rho)$ is continuous by the theorem of the maximum, the *ex-ante* planner's problem has a solution.

Next, we show that this solution can be characterized by simple first-order conditions. First, standard arguments show that the *interim* problem is solved by:

$$q_h = D(1,\xi), \quad q_\ell = D(0,\xi), \text{ and } q_m = D(\mu,\xi), \tag{80}$$

where $D(\mu,\xi)$ is a demand function defined exactly as in the main body of the paper, and ξ solves:

$$\rho\left[\mu D(1,\xi) + (1-\mu)D(0,\xi)\right] + (1-\rho)D(\mu,\xi) = s. \tag{81}$$

Now consider $W(\rho)$, the social value of choosing effort, at the *ex-ante* stage. Our main result is:

Proposition 12. *The planner's problem is solved by the unique $\rho^\star$ such that*

$$W'(\rho^\star) \leq 0 \text{ if } \rho^\star = 0, W'(\rho^\star) = 0 \text{ if } \rho^\star \in (0,1), \text{ and } W'(\rho^\star) \geq 0 \text{ if } \rho^\star = 1, \tag{82}$$

where

$$\begin{aligned} W'(\rho) = \left[\mu v(\theta_h, q_h) + (1-\mu)v(\theta_\ell, q_\ell)\right] - \left[\mu v(\theta_h, q_m) + (1-\mu)v(\theta_\ell, q_m)\right] \\ - \xi\left[\mu q_h + (1-\mu)q_\ell - q_m\right], \end{aligned} \tag{83}$$

and $\{q_s\}$ and ξ jointly solve (80) and (81) given ρ.

The expression for $W'(\rho)$ is obtained by an application of the envelope theorem. Clearly, condition (83) is necessary for optimality. To show uniqueness and sufficiency, we take another round of derivative to obtain that:

$$W''(\rho) = -\frac{d\xi}{d\rho}\left[\mu q_h + (1-\mu)q_\ell - q_m\right] = \frac{\left[\mu q_h + (1-\mu)q_\ell - q_m\right]^2}{\rho\left[\mu D_\xi(1,\xi) + (1-\mu)D_\xi(0,\xi)\right] + (1-\rho)D\xi(\mu,\xi)} < 0.$$

In the above, the first equality follows because, when $\{q_s\}$ are given by (79), then marginal utilities are equal to ξ. The second equality follows by calculating $d\xi/d\rho$ explicitly using the implicit function theorem. □

Finally, we argue that our restriction that banks choose the same level of effort is without loss of generality. Notice indeed that, with heterogeneous ρ, the social welfare in the interim stage, W, only depends on the average ρ. Given convexity of the cost function, the planner strictly prefers to have all banks choose a common level of effort.

B.4.3 Equilibrium

We now study the equilibrium choice of information collection effort and show that it coincides with the social optimum. Suppose that other banks exert a level of information collection effort equal to

$\bar{\rho}$. As in the paper, the *interim* equilibrium is socially optimal given $\bar{\rho}$. This implies that the *interim* equilibrium price is the unique solution $\bar{\xi}$ of (81), and the asset holdings are given by (80). *Ex-ante*, each individual bank chooses its level of information collection effort, ρ, taking as given the information collection of others, $\bar{\rho}$, which determines the *interim* equilibrium price, $\bar{\xi}$. To an individual bank, the value of choosing ρ is:

$$V(\rho \,|\, \bar{\xi}) = \max_{\{q_s\}} \rho \left[\mu v(\theta_h, q_h) + (1-\mu) v(\theta_\ell, q_\ell)\right] + (1-\rho)\left[\mu v(\theta_h, q_m) + (1-\mu) v(\theta_\ell, q_m)\right] \\ - \bar{\xi}\left\{\rho\left[\mu q_h + (1-\mu) q_\ell\right] + (1-\rho) q_m\right\}.$$

A bank's *ex-ante* effort choice problem is:

$$\max_{\rho \in [0,1]} V(\rho \,|\, \bar{\xi}) - C(\rho).$$

An *ex-ante* equilibrium is defined as a pair $(\bar{\rho}, \bar{\xi})$ such that: (i) $\bar{\xi}$ is an *interim* equilibrium price given $\bar{\rho}$, and (ii) $\bar{\rho}$ solves the bank's *ex-ante* effort choice problem given $\bar{\xi}$. Our main result is:

Proposition 13. *There exists a unique ex-ante equilibrium. In this equilibrium, bank's effort collection choice is socially optimal, i.e.,* $\bar{\rho} = \rho^\star$.

To show this proposition, we first use the envelope theorem to assert that:

$$V'(\rho \,|\, \bar{\xi}) = \left[\mu v(\theta_h, q_h) + (1-\mu) v(\theta_\ell, q_\ell)\right] - \left[\mu v(\theta_h, q_m) + (1-\mu) v(\theta_\ell, q_m)\right] \\ - \bar{\xi}\left\{\mu q_h + (1-\mu) q_\ell - q_m\right\},$$

where $\{q_s\}$ solves (80) given $\bar{\xi}$. Since $\{q_s\}$ only depend on $\bar{\xi}$, which a bank takes as given, we have that $V''(\rho \,|\, \bar{\xi}) = 0$. Since the cost function $C(\rho)$ is strictly convex, it thus follows that the *ex-ante* effort choice problem is strictly concave, and its solution is uniquely characterized by the first-order condition. Clearly, one sees that the equilibrium condition coincides with the optimality condition of the planning problem. □

Notice again that we need not worry about asymmetric equilibria in which banks choose heterogenous levels of efforts: given the price that will prevail at the interim stage, a bank's effort choice problem is strictly concave, so it has a unique maximizer.

To illustrate the proposition we consider the following numerical example. We use iso-elastic preferences $v(\theta, q) = \theta q^{1-\sigma}/(1-\sigma)$, and we set $\sigma = 0.5$, $s = 0.5$, $\theta_h = 1$, and $\theta_\ell = 0.1$. We assume that $\mu = 0.5$ and that the cost of effort is:

$$C(\rho) = c\frac{\rho^{1+\gamma}}{1+\gamma},$$

where $\gamma = 0.1$ and the constant c is chosen so that the planner's problem is maximized at $\rho^\star = 0.5$. In Figure 6, the social value of information collection effort, $W(\rho) - C(\rho)$, is shown as the plain red

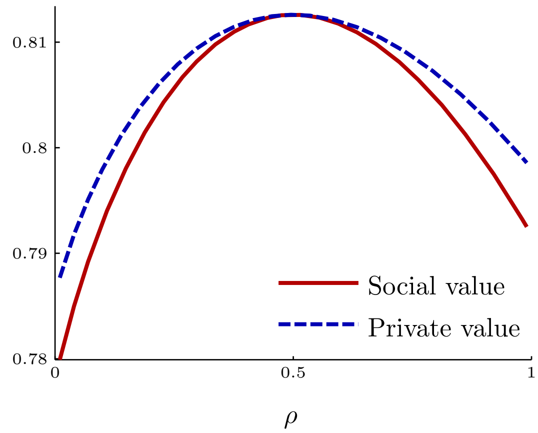

Figure 6: The social value (plain red) and private value (dashed blue) of information collection effort.

curve. The individual bank's private value of recovery effort given the equilibrium price $\xi^\star$ generated by $\rho^\star$, $V(\rho\,|\xi^\star) - C(\rho)$, is the dashed blue curve. One sees that the social value of effort differs from the social value. In particular, the social value is more concave than the private value: this is because the planner's value takes into account the impact of changing ρ on the (shadow) price of the asset, ξ, while an individual bank does not. However, one sees that the envelope theorem ensures that the private and social value coincide and are tangent to each other at $\rho = \rho^\star$.

B.5 Finite number of traders

In this appendix we offer some numerical calculations of an equilibrium when there is a finite number of traders, with and without preference uncertainty. We describe the evolution of traders' asset holdings and of the holding cost. Our calculations reveal that our main excess volume result continues to hold when there is a finite number of traders. In addition, since idiosyncratic preference shocks and updating times no longer average out, the model features a new source of holding cost volatility. Our calculations suggests that, relative to the known preference case with the same finite number of traders, preference uncertainty tends to mitigate this new source of volatility.

We consider a finite number N of traders but otherwise keep the model exactly as in the text. In particular, we continue to assume that traders behave competitively, as price takers. Studying price impact, along the line of Vayanos (1999) or Rostek and Weretka (2011) would introduce additional technical difficulties that go beyond the main objective of this appendix. Under price taking, the demand of trader $i \in \{1, \ldots, N\}$ at time u remains equal to $D(\pi_{\tau_u^i,u}, \xi_u)$, where τ_u^i denotes the last updating time of trader $i \in \{1, \ldots, N\}$ before the current time, u. What is different is the market clearing condition, which becomes:

$$\frac{1}{N}\sum_{i=1}^{N} D(\pi_{\tau_u^i,u}, \xi_u) = s.$$

One sees that, each trader's updating time before recovery becomes an aggregate shock: it changes that trader's demand and thus moves the price discretely.

Figure 7 shows the equilibrium holdings along a particular sample path of preference shocks and updating times. The number of traders is set to $N = 5$ and otherwise the parameters are the same as in our main parametric calculations. Equilibrium objects under preference uncertainty and known preferences are depicted by plain blue lines and dashed red lines, respectively. One sees clearly from the figure that the updating times of others become aggregate shocks and cause every trader to change its holdings. This is an additional source of trading volume, above and beyond the one identified in the continuum-of-traders case.

Figure 8 shows the cumulative volume along this particular sample path of shocks (left panel), as well as the average volume across $10,000$ sample paths (right panel). Both figures indicate that, just as in our main model, cumulative volume is larger with preference uncertainty than with known preferences.

Figure 9 shows the holding cost for the same particular sample path of shocks (left panel) as well as the average holding cost across 10,000 sample paths of shocks. One sees clearly from both panels that preference uncertainty tends to raise the holding cost at the inception of the liquidity shock, because traders who still have a low valuation believe they may have switched to a high valuation. One also sees that the full recovery is delayed, as traders need to wait for an updating time before being certain that they have a high valuation.

Finally, one may wonder what is the impact of having a finite number of traders on holding cost

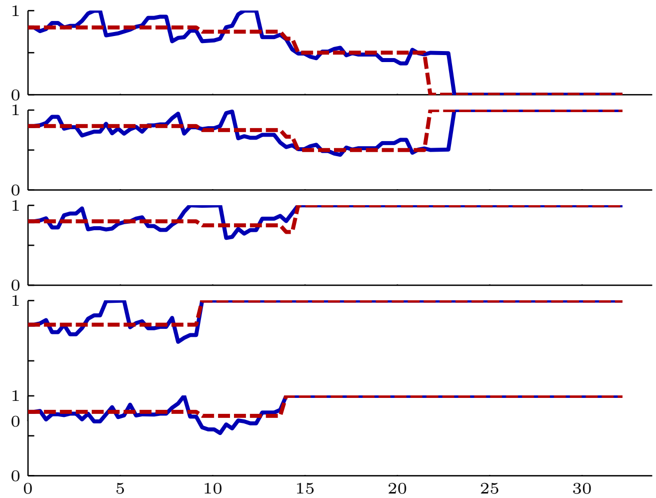

Figure 7: Holdings of 5 traders along a sample path of shocks, for known preferences (dashed red) vs. uncertain preferences (plain blue).

volatility, with and without preference uncertainty. One sees intuitively that, with known preferences, there are larger upward changes in holding costs. With preference uncertainty, there are many small changes in holding costs, upward and downward. Figure 10 confirms this observation by calculating the volatility of the percentage difference between the holding cost and the average holding cost across 10,000 simulations. The volatility with known preferences is higher, and peaks sooner, reflecting the large change in holding cost arising when sufficiently many traders have switched to high.

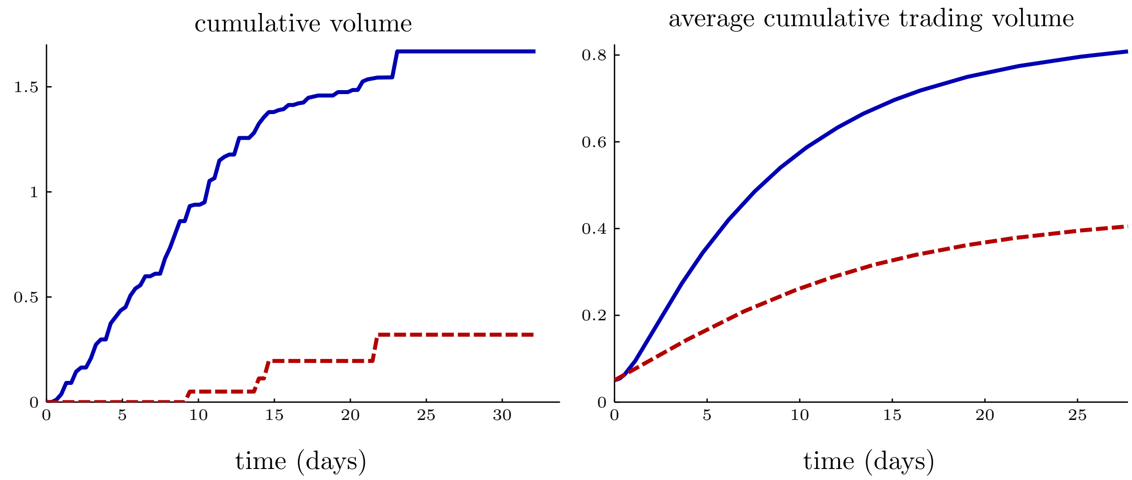

Figure 8: Cumulative trading volume along a sample path of shocks (left panel) and average cumulative trading volume across 10,000 sample paths of shocks, for known preferences (dashed red) vs. uncertain preferences (plain blue).

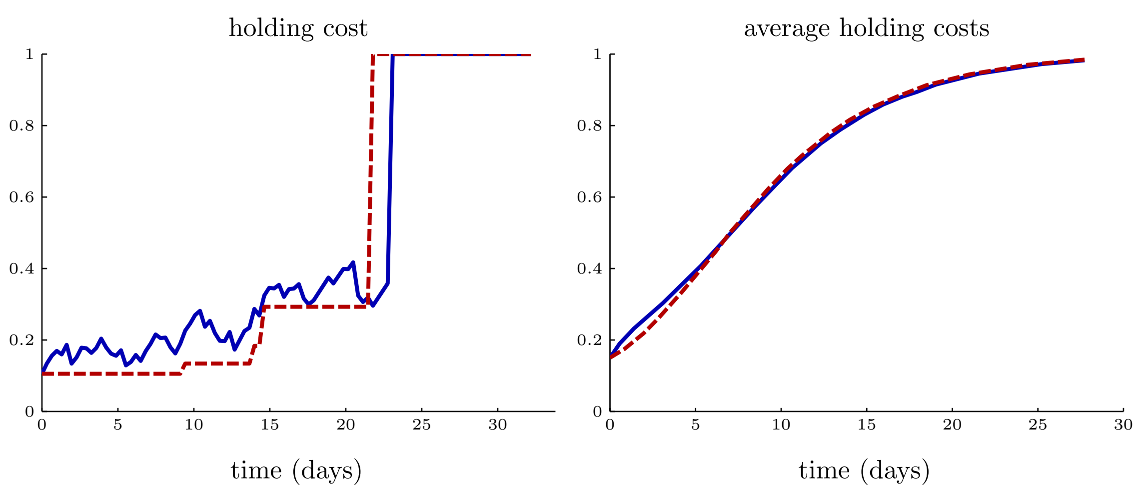

Figure 9: Holding cost along a sample path of shocks (left panel) and average holding cost across 10,000 sample paths of shocks, for known preferences (dashed red) vs. uncertain preferences (plain blue).

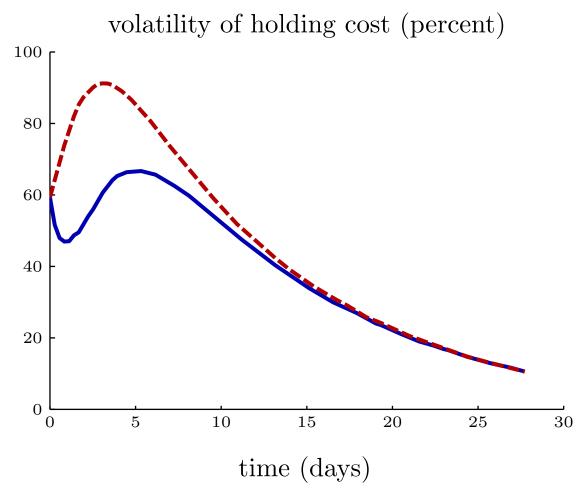

Figure 10: Volatility of holding costs across 10,000 sample paths of shocks, for known preferences (dashed red) vs. uncertain preferences (plain blue).