Web Appendix for 'Upstream Competition between Vertically Integrated Firms'

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This document is organized as follows. In Section I, we solve the model with downstream cost differentials under a linear specification of the demand functions. Section I.1 completely characterizes the equilibria of our model under Assumption 2 (Proposition 5). In Section I.2, we no longer make Assumption 2, and we solve for the sets of parameters' values such that monopoly-like (resp., complete foreclosure) equilibria exist (Proposition 11). We then investigate how these results are affected when downstream costs are convex, or when integrated firms are allowed to offer two-part tariff contracts on the upstream market. Section II solves the spatial competition model with exogenous locations (Proposition 12). Section III extends Proposition 5 to a more general framework with M integrated firms and N downstream firms (Proposition 10). Finally, Section IV proves that partial foreclosure equilibria degrade consumer surplus and social welfare, as stated in footnote 17 of the paper.

Most of these proofs are analytical, however, they tend to rely on very tedious calculations. For the sake of readability, we relegate most of these calculations in separate mathematica files, available at `http://sites.google.com/site/nicolasschutz/research`.

I Linear Demands and Downstream Cost Differentials[1]

I.1 Proof of Proposition 5

The proof follows the same logic as the proof of Proposition 4. As mentioned at the end of Section A.5 of the paper, we will also check that, when $\delta = 0$, the upstream supplier never wants to undercut discontinuously to drive firm d out of the market.

Determine all the equilibria. To begin with, we normalize all upstream and downstream costs to $c = c_u = 0$. Consider that integrated firm i supplies the upstream market at price a_i

[1] All the calculations for this section can be found in Mathematica file `Linear-demands-cost-differentials.nb`

and denote its integrated rival by j. The best-response functions are uniquely defined since, for all downstream and upstream prices and for $k \in \{1, 2, d\}$, we have $\frac{\partial^2 \tilde{\pi}_k^{(i)}}{\partial p_k^2} = -2(1+\frac{2}{3}\gamma) < 0$. The stability condition is satisfied, since, for all $k \neq k'$, we have $\left|\frac{\partial BR_k^{(i)}}{\partial p_{k'}}\right| = \frac{\gamma}{6+4\gamma} < 1$. There is a unique downstream equilibrium, which can be computed by solving the set of first-order conditions. The equilibrium quantity served by downstream firm d is positive if and only if a_i is below a threshold $a_{\max}(\gamma, \delta)$ that we compute in the Mathematica file, and $\pi_d^{(i)}(a_i)$ is decreasing for $a_i \leq a_{\max}(\gamma, \delta)$; hence Assumption 1 is satisfied. The profit of the upstream supplier $\pi_i^{(i)}(a_i)$ is strictly quasi-concave and its maximum is reached for $a_i = a_m(\gamma, \delta) \in (0, a_{\max}(\gamma, \delta))$; hence Assumption 3 is satisfied.

$\pi_i^{(i)}(a_i)$ and $\pi_j^{(i)}(a_i)$ are parabolas, they cross each other twice, in $a_i = 0$ and in $a_i = a_*(\gamma, \delta) > 0$. $\pi_i^{(i)}(a_i)$ is strictly concave and $\pi_j^{(i)}(a_i)$ is strictly convex, therefore $\pi_i^{(i)}(a_i) \geq \pi_j^{(i)}(a_i)$ if and only if $a_i \in [0, a_*(\gamma, \delta)]$. Then, we show that for all γ there exists $\overline{\delta}(\gamma)$ such that $a_m(\gamma, \delta) \geq a_*(\gamma, \delta)$ if and only if $\delta \geq \overline{\delta}(\gamma)$. When this condition is satisfied, we have the same equilibrium outcomes on the upstream market as in Proposition 4: the perfect competition outcome $a_1 = a_2 = 0$, the matching-like outcome $a_1 = a_2 = a_*(\gamma, \delta)$, the monopoly-like outcome in which $a_1 = a_m(\gamma, \delta)$ and firm 2 makes no upstream offer, and the monopoly-like outcome in which $a_2 = a_m(\gamma, \delta)$ and firm 1 makes no upstream offer. Conversely, if condition $\delta \geq \overline{\delta}(\gamma)$ is not satisfied, then, only the perfect competition outcome is an equilibrium.

Restrictions on the range of parameters. Intuitively, if the entrant is highly inefficient (large positive δ) then it cannot be active on the downstream market. Formally, we show in the Mathematica file that, for all γ, there exists a threshold value $\delta_{sup}(\gamma)$ such that firm d obtains strictly positive profits when $a_i = a_m(\gamma, \delta)$ if and only if $\delta < \delta_{sup}(\gamma)$. Moreover $\delta_{sup}(\gamma) > \max\{0, \overline{\delta}(\gamma)\}$ for all γ.

Similarly, if the entrant is highly efficient (large negative δ) then the integrated firms will not be active on the downstream market. We show that, for all γ, there exists $\delta_{inf}(\gamma)$ such that integrated firms make strictly positive profits when $a_i = 0$ if and only if $\delta > \delta_{inf}(\gamma)$. Moreover $\delta_{inf}(\gamma) < \min\{0, \overline{\delta}(\gamma)\}$ for all γ.

Therefore, the above analysis is valid in the range of parameter values $\{(\gamma, \delta) : \gamma \geq 0, \delta_{inf}(\gamma) < \delta < \delta_{sup}(\gamma)\}$.

No discontinuous undercutting. Now we check that, at the equilibrium of the downstream competition subgame, no firm is willing to "undercut discontinuously", i.e., to set a downstream price so low that it would exclude (at least) one of the competitors. To see why we have to check this point, we define $P_k(k')$ and $P_k(k'')$, $\{k, k', k''\} = \{1, 2, d\}$, as the threshold values of p_k such that competitor k' (resp. k'') is driven out of the downstream market when p_k decreases below $P_k(k')$ (resp. $P_k(k'')$). Notice first that the profit function $\tilde{\pi}_k^{(i)}$ is continuous in p_k since the downstream demand D_k is also continuous. However, $\tilde{\pi}_k^{(i)}$ is kinked at points $p_k = P_k(k')$ and $p_k = P_k(k'')$. Because of these kinks, $\tilde{\pi}_k^{(i)}(p_k)$ may not be strictly concave, even though it is a piecewise quadratic polynomial function.

In fact we can see that firms j and d never want to undercut discontinuously, and that the upstream supplier i never wants to undercut discontinuously the integrated rival j. The intuition is that when a firm lowers its downstream price to the point where it eliminates a downstream competitor, its demand function becomes less elastic to its own price. Formally, for two different firms k and k' such that $(k, k') \neq (i, d)$, $\partial D_k/\partial p_k < 0$ jumps upwards when p_k goes below $P_k(k')$. This implies that $\partial\tilde{\pi}_k^{(i)}/\partial p_k = D_k + p_k \partial D_k/\partial p_k$ also jumps upwards when p_k goes below $P_k(k')$. Since $\tilde{\pi}_k^{(i)}(p_k)$ is strictly increasing in the right neighborhood of $P_k(k')$ (since $P_k(k') < p_k^{(i)}(a_i)$), then it is even more increasing in the left neighborhood of $P_k(k')$. Therefore the profit function $\tilde{\pi}_k^{(i)}(p_k)$ is globally strictly concave.

By contrast, profit function $\tilde{\pi}_i^{(i)}(p_i)$ may not be strictly concave around the kink in $p_i = P_i(d)$. The reason is that the profit function of the upstream supplier also includes the upstream profit term $a_i D_d$, whose derivative jumps downwards (from a strictly positive value to zero) when p_i goes below $P_i(d)$. Therefore the profit function $\partial\tilde{\pi}_i^{(i)}/\partial p_i$ may jump upwards or downwards at point $p_i = P_i(d)$. In fact we do not necessarily need the jump to be upward. A sufficient condition for undercutting discontinuously to be unprofitable is that $\tilde{\pi}_i^{(i)}(p_i)$ remains upward slopping when p_i gets below $P_i(d)$.[2] In the Mathematica file, we show that there exists a threshold value $a_{Undercut}(\gamma, \delta)$ such that this condition is satisfied if and only if $a_i \leq a_{Undercut}(\gamma, \delta)$. So all we need to check is that $a_m(\gamma, \delta) < a_{Undercut}(\gamma, \delta)$. This condition, in turn, is equivalent to δ being below a threshold value $\delta_{Undercut}(\gamma)$, which is strictly larger than $\overline{\delta}(\gamma)$ for all γ. Therefore, by restricting the set of parameter values (γ, δ) such that $\delta < \delta_{Undercut}(\gamma)$, Assumption 2 is satisfied and our analysis leading to Proposition 5 is valid.

Finally, we note that $\delta_{Undercut}(\gamma) > 0$ for all γ. This ensures that discontinuous undercutting is never an issue in the proof of Proposition 4 ($\delta = 0$), as claimed at the end of Section A.5 in the paper.

I.2 Complete vs. Partial Foreclosure

I.2.1 Proof of Proposition 11

This section comments the Mathematica file leading to Proposition 11.

We start by computing the equilibrium profits of the integrated firms, $\pi_i^{(\varnothing)}$, $i \in \{1, 2\}$, when firm d does not obtain the input. There is a complete foreclosure equilibrium if and

[2] This amounts to requiring strict quasiconcavity instead of strict concavity.

only if $\pi_i^{(\varnothing)} \geq \pi_i^{(i)}(a_m)$. We show that, for all γ, there exists $\overline{\overline{\delta}}(\gamma)$ such that the complete foreclosure condition is equivalent to $\delta \geq \overline{\overline{\delta}}(\gamma)$.[3]

Therefore there is a complete foreclosure equilibrium when $\delta \geq \overline{\overline{\delta}}(\gamma)$; there are monopoly-like equilibria when $\overline{\delta}(\gamma) \leq \delta \leq \overline{\overline{\delta}}(\gamma)$; the Bertrand equilibrium is the only equilibrium when $\delta \leq \min\{\overline{\delta}(\gamma), \overline{\overline{\delta}}(\gamma)\}$. It remains to compare the threshold values $\overline{\delta}(\gamma)$ and $\overline{\overline{\delta}}(\gamma)$. We find that there exists $\overline{\overline{\gamma}}$ such that $\overline{\delta}(\gamma) \leq \overline{\overline{\delta}}(\gamma)$ if and only if $\gamma \geq \overline{\overline{\gamma}}$, which concludes the proof of Proposition 11.

I.2.2 Complete vs. Partial Foreclosure with Two-Part Tariffs

The Mathematica file also includes computations for the case in which integrated firms use two-part tariffs on the upstream market.

We first compute $a_{tp} = \arg\max_{a_i} \pi_i^{(i)}(a_i) + \pi_d^{(i)}(a_i)$. As explained in the paper in the extension on two-part tariffs, when complete foreclosure is ruled out, there is always partial foreclosure. The partial foreclosure equilibrium is monopoly-like if and only if $\pi_i^{(i)}(a_{tp}) + \pi_d^{(i)}(a_{tp}) \leq \pi_j^{(i)}(a_{tp})$; we show that there exists a threshold $\overline{\delta}_{tp}(\gamma)$ such that this condition is equivalent to $\delta \geq \overline{\delta}_{tp}(\gamma)$; otherwise the partial foreclosure equilibrium is matching-like.

Once we relax Assumption 2, there is a complete foreclosure equilibrium if and only if the duopoly profit $\pi_i^{(\varnothing)}$ is larger than $\pi_i^{(i)}(a_{tp}) + \pi_d^{(i)}(a_{tp})$. We show that there exists a threshold $\overline{\overline{\delta}}_{tp}(\gamma)$ such that this condition is equivalent to $\delta \geq \overline{\overline{\delta}}_{tp}(\gamma)$.

It remains to compare the two thresholds. We find that $\overline{\overline{\delta}}_{tp}(\gamma) \leq \overline{\delta}_{tp}(\gamma)$ for all γ. Therefore, there is a complete foreclosure equilibrium when $\delta \geq \overline{\overline{\delta}}_{tp}(\gamma)$, and a symmetric partial foreclosure equilibrium when $\delta \leq \overline{\overline{\delta}}_{tp}(\gamma)$.

I.2.3 Complete vs. Partial Foreclosure with Convex Downstream Costs

With a convex quadratic term $c_q D_k^2$, $c_q > 0$, added to the linear downstream cost, we solve for the equilibrium exactly as before. The additional difficulty is that we no longer obtain closed form expressions for the thresholds $\overline{\delta}$ and $\overline{\overline{\delta}}$. Instead we have to find them numerically. The rest of the resolution works just as before. When we increase c_q starting from 0, we find that $\overline{\overline{\gamma}}$ decreases and that the size of the partial foreclosure region expands.

[3] $\overline{\overline{\delta}}(\gamma)$ is above $\delta_{inf}(\gamma)$ and below $\delta_{Undercut}(\gamma)$ for all γ.

II Proof of Proposition 12[4]

In this section, we solve the spatial competition model with exogenous locations. A consumer purchasing from a firm located at distance x pays transport cost tx^2. We normalize t to 1, and upstream and downstream costs to zero without loss of generality. Solving for the marginal consumers' locations, we get the triopoly demand functions:

$$\begin{aligned} D_1 &= \frac{1}{6} - \frac{3}{2}(p_1 - p_2) + \frac{d}{2} - \frac{p_1 - p_d}{2d} \\ D_2 &= \frac{1}{6} - \frac{3}{2}(p_2 - p_1) + \frac{2/3 - d}{2} - \frac{p_2 - p_d}{2(2/3 - d)} \\ D_d &= \frac{1}{3} - \frac{p_d - p_1}{2d} - \frac{p_d - p_2}{2(2/3 - d)} \end{aligned}$$

and the duopoly demand functions:

$$\begin{aligned} D_1^{duo} &= \frac{1}{2} - \frac{9}{4}(p_1 - p_2) \\ D_2^{duo} &= \frac{1}{2} - \frac{9}{4}(p_2 - p_1) \end{aligned}$$

Consider that firm 1 is supplying the upstream market at price a_1. There is a unique Nash equilibrium on the downstream market, which we compute by solving for the first-order conditions. Plugging these equilibrium prices into firms' profit functions, we find that firm 1's profit, $\pi_1^{(1)}(a_1)$, is strictly concave in a_1, and reaches its maximum at price $a_m^1(d)$.[5] We then check that, for all $0 < d < 2/3$, firm d makes positive profits when its purchases the input from firm 1 at price $a_m^1(d)$.

Using the duopoly demand functions, we solve for firms 1 and 2's Nash equilibrium prices when firm d is completely foreclosed. We can then compute each firm's duopoly profit, which we denote by Π^{duo}.

There exists a complete foreclosure equilibrium if and only if $\Pi^{duo} \geq \max\{\pi_1^{(1)}(a_m^1(d)), \pi_2^{(2)}(a_m^2(d))\}$. There exists a cutoff $0 < d_{cutoff} < 2/3$, such that $\pi_1^{(1)}(a_m^1(d)) > \Pi_1^{Duo}$ if and only if $0 < d < d_{cutoff}$, and $\pi_1^{(1)}(a_m^1(d)) < \Pi_1^{Duo}$ if and only if $d_{cutoff} < d < 2/3$. Since $d_{cutoff} \simeq 0.285$, and by symmetry between firms 1 and 2, this implies that there is a complete foreclosure equilibrium if and only if $d \in [d_{cutoff}, 2/3 - d_{cutoff}]$. This also implies that there is no monopoly-like equilibrium in which firm 1 (resp., firm 2) supplies the upstream market if $d > d_{cutoff}$ (resp., if $d < 2/3 - d_{cutoff}$).

[4] All the calculations for this section can be found in Mathematica file `Proof-prop-12.nb`

[5] Notice that, since integrated firms are asymmetrically located, their monopoly upstream prices may be different.

Now, we prove that, for all $0 < d < d_{cutoff}$, there exists a monopoly-like equilibrium in which firm 1 supplies the upstream market. Assume firm 1 sells the input at price $a_m^1(d)$. Since $0 < d < d_{cutoff}$, firm 1 does not want to exit the upstream market. If firm 2 undercuts and captures the upstream market, it cannot get more than $\pi_2^{(2)}(a_m^2(d))$, by definition of $a_m^2(d)$. Using Mathematica, we find that $\pi_2^{(2)}(a_m^2(d)) < \pi_2^{(1)}(a_m^1(d))$ for all $0 < d < d_{cutoff}$, therefore, it is never profitable for firm 2 to undercut on the upstream market.

By symmetry, this also implies that for $2/3 - d_{cutoff} < d < 2/3$, there is a monopoly-like equilibrium in which firm 2 is the upstream supplier.

Discontinuous undercutting. We know from the proof of Proposition 5 that the upstream supplier may have incentives to undercut discontinuously on the downstream market so as to exclude the unintegrated downstream firm. In the following, we prove the existence of a threshold $0 < d_{undercut} < d_{cutoff}$, such that:

- If $d_{undercut} \leq d < d_{cutoff}$, then, for all $a_1 \in [0, a_m^1(\delta)]$, firm 1 has no incentives to undercut discontinuously. This implies that the monopoly-like equilibrium we derived before is, indeed, an equilibrium.

- If $0 < d < d_{undercut}$, then, there exists a threshold $a_{undercut}(d)\ in(0, a_m^1(d))$, such that the upstream supplier wants to undercut discontinuously on the downstream market if and only if $a_{undercut}(d) < a_1 \leq a_m^1(d)$. This implies that the monopoly-like equilibrium we derived before is, in fact, not an equilibrium, because the unintegrated downstream firm cannot be active when it purchases the input from firm 1 at price $a_m^1(d)$. In this case, we show that firm 1's monopoly upstream price has to be redefined as $a_{undercut}(d)$, and that, with this properly defined monopoly price, there exists a monopoly-like equilibrium where firm 1 supplies the upstream market.

Assume $0 < d < d_{cutoff}$, and suppose firm 1 supplies the upstream market at price $a_1 \in [0, a_m^1(d)]$. As in the proof of Proposition 5, we note that excluding one firm by undercutting discontinuously makes a firm's demand less price-sensitive. This ensures that firms 2 and d's profit functions are strictly concave, and therefore, that these firms have no incentives to undercut discontinuously on the downstream market.

Now, consider firm 1. Using Mathematica, we show that there exists two cutoffs, $P_{12} < P_{1d}$, such that, if firms 2 and d set downstream equilibrium prices $p_2^{(1)}(a_1)$ and $p_d^{(1)}(a_1)$, then, firm 2 (resp., firm d) supplies a positive quantity if and only if $p_1 > P_{12}$ (resp., $p_1 > P_{1d}$). Using the same argument as in the previous paragraph, it is clear that firm 1's profit is strictly concave on interval $(-\infty, P_{1d}]$. Let us define the following function:

$$\hat{\pi}(p_1) : p \in \mathbb{R} \mapsto p_1 \left(\frac{1}{2} - \frac{9}{4}(p_1 - p_2^{(1)}(a_1)) \right).$$

Notice that $\hat{\pi}(.)$ is strictly concave on $\mathbb{R}$, and that $\hat{\pi}(p_1) = \tilde{\pi}_1^{(1)}(p_1, p_2^{(1)}(a_1), p_d^{(1)}(a_1), a_1)$ for all $p_1 \in [P_{12}, P_{1d}]$. We compute $P_1^{dev} \equiv \arg\max_{p_1 \in \mathbb{R}} \hat{\pi}(p_1)$. We show that, for all $d \in (0, d_{cutoff})$, there exists a threshold $a_{NotConcave}(d) \in (0, a_m^1(d))$, such that:

- If $0 \leq a_1 \leq a_{NotConcave}(d)$, then, $P_1^{dev} \geq P_{1d}$. By concavity, this implies that $\hat{\pi}(.)$ is strictly increasing on interval $[P_{12}, P_{1d}]$. Since firm 1's profit is concave on $(-\infty, P_{1d}]$, it follows that firm 1's profit is increasing on this interval, and strictly quasi-concave on the real line. In this case, undercutting discontinuously is not profitable for firm 1.

- If $a_{NotConcave}(d) < a_1 \leq a_m^1(d)$, then, $P_{12} < P_1^{dev} < P_{1d}$. In this case, firm 1's profit function is not quasi-concave, as it has exactly two local maxima: P_1^{dev} and $p_1^{(1)}(a_1)$.

Now, assume that $a_{NotConcave}(d) < a_1 \leq a_m^1(d)$, and let us check whether firm 1 wants to set P_1^{dev} instead of $p_1^{(1)}(a_1)$. Denote by Π_1^{dev} firm 1's profit if it sets P_1^{dev}. Using Mathematica, we prove the existence of a threshold $0 < d_{undercut} < d_{cutoff}$, such that:

- If $d \in [d_{undercut}, d_{cutoff})$, then, $\Pi_1^{dev} \leq \pi_1^{(1)}(a_1)$ for all $a_1 \in [0, a_m^1(d)]$. This implies, in particular, that undercutting discontinuously is not profitable for firm 1 when $a_1 = a_m^1(d)$. When $d \in [d_{undercut}, d_{cutoff})$, our monopoly-like equilibrium is, indeed an equilibrium.

- If $d \in (0, d_{undercut})$, then, there exists a threshold $a_{undercut}(d) \in (a_{NotConcave}(d), a_m^1(d))$, such that firm 1 is better off setting P_1^{dev} if and only if $a_1 > a_{undercut}(d)$.

In the latter case, when $a_1 = a_m^1(d)$, firm 1 wants to undercut discontinuously on the downstream market, and therefore, the monopoly-like equilibrium we derived before is, in fact, not an equilibrium. However, we claim that, when $d \in (0, d_{undercut})$, firm 1's monopoly upstream price is not $a_m^1(d)$, but rather $a_{undercut}(d)$. To see this, notice that any offer strictly above $a_{undercut}(d)$ is not acceptable, since firm d cannot be active with such an input price. Besides, firm 1's profit is strictly increasing up to $a_1 = a_{undercut}(d)$. As a result, if firm 1 wants to make an acceptable offer, the best thing it can do is propose $a_1 = a_{undercut}(d)$. We therefore redefine the "true" monopoly upstream price as $a_m^1(d) \equiv a_{undercut}(d)$ for all $d \in (0, d_{undercut})$.

We can then check that $\pi_1^{(1)}(a_{undercut}(d)) > \Pi^{duo}$ and $\pi_2^{(1)}(a_{undercut}(d)) > \pi_2^{(2)}(a_m^2(d))$ for all $d \in (0, d_{undercut})$, and we conclude that monopoly-like equilibria also exist for these values of parameter d.

Input differentiation. Under input differentiation, the unintegrated downstream firm is located at distance d from its upstream supplier. Define $a_m(d) \equiv a_m^{(1)}(d)$ for all $0 < d \leq 1/3$. Using Mathematica, we show that:

- $\Pi^{duo} \geq \pi_1^{(1)}(a_m(d))$, i.e., complete foreclosure is an equilibrium, if and only if $d_{cutoff} \leq d \leq 1/3$.
- $\Pi^{duo} \leq \pi_1^{(1)}(a_m(d)) \leq \pi_2^{(1)}(a_m(d))$, i.e., there are monopoly-like equilibria, if and only if $0 < d \leq d_{cutoff}$.

As before, it is profitable for the upstream supplier to undercut discontinuously on the downstream market if and only if $d < d_{undercut}$. We check that, for all $0 < d < d_{undercut}$, $\Pi^{duo} \leq \pi_1^{(1)}(a_{undercut}(d)) \leq \pi_2^{(1)}(a_{undercut}(d))$, and we conclude that monopoly-like equilibria also exist for these values of parameter d.

III Proof of Proposition 10[6]

This section proves the analog of Proposition 5 in a more general framework with M integrated firms and N downstream firms.

Necessary and sufficient condition for monopoly-like equilibria to exist. The proof follows the same steps as the proof of Proposition 5. As before, we can normalize c_u and c to zero without loss of generality. To begin with, we derive the demand functions by solving the representative consumer's program. Denote by V the set of varieties of the downstream product. By assumption, there are $|V| = M + N$ such varieties, where $|V|$ denotes the cardinality of set V. Define also $\tilde{V}$ as the set of varieties that are actually purchased by the representative consumer, i.e., loosely speaking, the varieties whose prices are not too high. Then, it is straightforward to show that the demand for variety $k \in \tilde{V}$ is given by:

$$D_k = \frac{1+\gamma}{\gamma|\tilde{V}| + |V|}\left(1 - p_k + \frac{\gamma|\tilde{V}|}{|V|}\left(\frac{\sum_{k' \in \tilde{V}} p_{k'}}{|\tilde{V}|} - p_k\right)\right) \tag{1}$$

Assume firm 1 is supplying all unintegrated downstream firms at price a_1. We focus on symmetric Nash equilibria on the downstream market. We are looking for triples (p_1, p_2, p_d) such that the upstream supplier charges p_1, the integrated rivals charge p_2, the unintegrated downstream firms charge p_d, and no firm wants to deviate. Taking first-order conditions, and

[6] All the calculations for this section can be found in Mathematica file `Proof-prop-10.nb`

using the symmetric Nash equilibrium assumption, it follows that (p_1, p_2, p_d) has to satisfy the following three equations:

$$\begin{aligned}
0 &= 1 - p_1 + \gamma\left(\frac{p_1 + (M-1)p_2 + Np_d}{M+N} - p_1\right) - p_1\left(1 + \gamma(1 - \frac{1}{M+N})\right) + a_1\frac{\gamma N}{M+N} \\
0 &= 1 - p_2 + \gamma\left(\frac{p_1 + (M-1)p_2 + Np_d}{M+N} - p_2\right) - p_2\left(1 + \gamma(1 - \frac{1}{M+N})\right) \\
0 &= 1 - p_d + \gamma\left(\frac{p_d + (M-1)p_2 + Np_d}{M+N} - p_2\right) - (p_d - a_1 - \delta)\left(1 + \gamma(1 - \frac{1}{M+N})\right)
\end{aligned}$$

We then use Mathematica to show that this system has exactly one solution, i.e., there is a unique (symmetric) Nash equilibrium in the downstream competition subgame. We can then plug these equilibrium prices, (p_1^*, p_2^*, p_d^*), into the firms' profit functions. We obtain that the downstream firms supply a positive quantity and make positive profits if and only if a_1 is lower than a threshold $a_{max}^{M,N}(\gamma, \delta)$. We can also compute $a_m^{M,N}(\gamma, \delta)$, the monopoly upstream price.

Denote by $\Pi_{US}(a_1)$ the upstream supplier's profit, and by $\Pi_{IR}(a_1)$ one of its integrated rivals' profit. We show that $\Pi_{US}(a_1) - \Pi_{IR}(a_1)$ is a strictly concave and quadratic polynomial in a_1. Therefore, it is strictly positive between its roots, 0 and $a_*^{M,N}(\gamma, \delta)$.

Assume the upstream supplier, firm 1, sets its monopoly upstream price $a_m^{M,N}(\gamma, \delta)$. Clearly, by Assumptions 1 and 2, the upstream supplier does not want to deviate. For the integrated rivals, there are two possible deviations.

First, an integrated firm may choose to match the upstream supplier's offer. In this case, we assume that all downstream firms continue to purchase their inputs from firm 1. This is indeed a Nash equilibrium of the offer acceptance subgame, since, under Assumption 4, downstream firms elect their upstream providers after downstream prices have been set, and are therefore indifferent between two offers at the same price. Therefore, this kind of deviation cannot destabilize our monopoly-like outcome.

Second, an integrated firm may choose to undercut the upstream supplier. This deviation is not strictly profitable if and only if $\Pi_{US}(a_1) \leq \Pi_{IR}(a_1)$, which is equivalent to $a_*^{M,N}(\gamma, \delta) \leq a_m^{M,N}(\gamma, \delta)$. Using Mathematica, we show that, for all $\gamma \geq 0$, there exists a threshold $\overline{\delta}^{M,N}(\gamma)$ such that $a_*^{M,N}(\gamma, \delta) \leq a_m^{M,N}(\gamma, \delta)$ if and only if $\delta \geq \overline{\delta}^{M,N}(\gamma)$.

Restrictions on the range of parameters. As in the proof of Proposition 5, we define $\delta_{sup}^{M,N}(\gamma)$ as the cutoff such that downstream firms can make positive profits with input price $a_1 = a_m^{M,N}(\gamma, \delta)$ if and only if $\delta < \delta_{sup}^{M,N}(\gamma)$. Similarly, $\delta_{inf}^{M,N}(\gamma)$ is the threshold such that integrated firms make positive profits when input price is $a_1 = 0$ if and only if $\delta > \delta_{inf}^{M,N}(\gamma)$.

Using Mathematica, we compute these thresholds, and we show that $\delta_{inf}^{M,N}(\gamma) < \overline{\delta}^{M,N}(\gamma) < \delta_{sup}^{M,N}(\gamma)$ for all $\gamma \geq 0$. Therefore, the above analysis is valid in the range of parameter values $\{(\gamma, \delta, M, N) : \gamma \geq 0,\ \delta_{inf}(\gamma) < \delta < \delta_{sup}(\gamma),\ M \geq 2,\ N \geq 1\}$.

No discontinuous undercutting. As in the proof of Proposition 5, it is straightforward to show that the downstream firms and the integrated firms which do not supply the upstream market do not want to undercut discontinuously. As before, the reason is that excluding one (or several) competitors makes a firm's demand less price sensitive, which ensures that profit functions for all firms except the upstream supplier are strictly concave.

By contrast, undercutting discontinuously may be profitable for the upstream supplier. A sufficient condition for such a deviation to be unprofitable is that the upstream supplier's profit function be strictly quasi-concave. Define $P_1(d)$ as the (downstream) price threshold such that, starting from the Nash equilibrium prices, all downstream firms receive zero demand if the upstream supplier deviates and sets a price below $P_1(d)$. As argued in the proof of Proposition 5, the upstream supplier's profit is globally strictly quasi-concave if and only if the derivative of this function with respect to p_1 has a positive limit as p_1 approaches $P_1(d)$ from the left.

When p_1 is smaller than $P_1(d)$, firm 1's profit is just:

$$p_1 \frac{1+\gamma}{\gamma M + M + N}\left(1 - p_1 + \frac{\gamma M}{M+N}\left(\frac{p_1 + (M-1)p_2^*}{M} - p_1\right)\right)$$

In the Mathematica file, we compute $P_1(d)$ and differentiate the above expression with respect to p_1. We then compute the limit of this derivative as p_1 goes to $P_1(d)$, and show that it is positive if and only if a_1 is smaller than some threshold $a_{Undercut}^{M,N}(\gamma, \delta)$. Now, all we need to check is that $a_m^{M,N}(\gamma, \delta)$ is indeed below $a_{Undercut}^{M,N}(\gamma, \delta)$. We show that this is the case, provided that δ is smaller than some threshold $\delta_{Undercut}^{M,N}(\gamma)$. We also show that $\delta_{Undercut}^{M,N}(\gamma) > \overline{\delta}^{M,N}(\gamma)$. Therefore, by restricting ourselves to parameter values (γ, δ, M, N) such that $\delta < \delta_{Undercut}^{M,N}(\gamma)$, the analog of Assumption 3 is satisfied and our analysis leading to Proposition 10 is valid.

IV Welfare Analysis

We prove the following result:

Proposition. *Assume that downstream prices are strategic complements.*

- *Then, consumers strictly prefer the perfect competition outcome to a partial foreclosure equilibrium.*

- *Besides, if firms' downstream divisions are identical and downstream costs are weakly convex, then, social welfare is strictly higher in the perfect competition outcome than in a partial foreclosure equilibrium.*

Proof. Consider a partial foreclosure equilibrium with upstream price $\hat{a} > c_u$. Assume that downstream prices are strategic complements, and let us show that downstream prices are higher in the partial foreclosure equilibria than in the perfect competition outcome, and that at least one of these prices is strictly higher.

Strategic complementarity writes as $\partial^2 \tilde{\pi}_k^{(i)} / \partial p_k \partial p'_k \geq 0$ for $k \neq k'$, hence the game defined by payoff functions $(p_k, p_{-k}) \in [0, +\infty)^3 \mapsto \tilde{\pi}_k^{(i)}(p_k, p_{-k}, a)$ is smooth supermodular, parameterized by $a \in \{c_u, \hat{a}\}$. For all k, $\partial \tilde{\pi}_k^{(i)}(.,.,\hat{a}) / \partial p_k \geq \partial \tilde{\pi}_k^{(i)}(.,.,c_u) / \partial p_k$, therefore, $\tilde{\pi}_k^{(i)}(p_k, p_{-k}, a)$ has increasing differences in (p_k, a). Besides, the downstream equilibrium is, by assumption, unique. Supermodularity theory (see Vives 1999, Theorem 2.3) tells us that the equilibrium of this game is increasing in a. Therefore, $p_k^{(i)}(c_u) \leq p_k^{(i)}(\hat{a})$ for all k.

Besides, by Lemma 1, $p_i^{(i)}(\hat{a}) > p_j^{(i)}(\hat{a}) \geq p_j^{(i)}(c_u) = p_i^{(i)}(c_u)$, therefore, $p_i^{(i)}(\hat{a}) > p_i^{(i)}(c_u)$. Therefore, consumers are worse off in the partial foreclosure equilibrium.

Assume also: a representative consumer with a quasi-linear, continuously differentiable and quasi-concave utility function exists; firms have symmetric and identical demands; firms have the same convex downstream costs functions. Let us show that partial foreclosure lowers the social welfare.

When the upstream price is set at marginal cost, the three firms are perfectly identical. Hence, since the downstream equilibrium is unique, it is symmetric, and $p_k^{(i)}(c_u) = \hat{p}$ for all k. Let $(\hat{p}_1, \hat{p}_2, \hat{p}_3)$ the permutation of the triple $\left(p_k^{(i)}(\hat{a})\right)_{k=1,2,d}$ such that $\hat{p}_1 \leq \hat{p}_2 \leq \hat{p}_3$, and let us relabel firms so that firm k is the firm that charges $\hat{p}_k$ when the upstream price is $\hat{a}$. Recall that $\hat{p}_k \geq \hat{p}$ for all k, and that this inequality is strict for $k = 3$.

Keeping in mind that firms have been relabeled, let us denote by $U(q_0, q_1, q_2, q_3) = q_0 + u(q_1, q_2, q_3)$ the utility function of the representative consumer, where q_0 denotes consumption of the numeraire, and q_k denotes consumption of product $k \in \{1, 2, 3\}$. We can then write the social welfare as a function of the downstream price vector p:[7]

$$W(p) = u\left(D_1(p), D_2(p), D_3(p)\right) - \sum_{k=1}^{3} \left\{c_u D_k(p) + c\left(D_k(p)\right)\right\},$$

where $c(.)$ denotes the downstream cost function, which, by assumption, is the same for the three firms. This welfare function is continuously differentiable, since functions u, c and D_k are continuously differentiable. To prove the result, we need to show that $W(\hat{p}_1, \hat{p}_2, \hat{p}_3) -$

[7] This function does not depend on the upstream price.

$W(\hat{p},\hat{p},\hat{p})$, the variation in social welfare when downstream prices increase from $(\hat{p},\hat{p},\hat{p})$ to $(\hat{p}_1,\hat{p}_2,\hat{p}_3)$, is strictly negative. This variation can be written as:

$$\begin{aligned}(W(\hat{p}_1,\hat{p}_2,\hat{p}_3) - W(\hat{p}_1,\hat{p}_2,\hat{p}_2)) + (W(\hat{p}_1,\hat{p}_2,\hat{p}_2) - W(\hat{p}_1,\hat{p}_1,\hat{p}_1)) + (W(\hat{p}_1,\hat{p}_1,\hat{p}_1) - W(\hat{p},\hat{p},\hat{p}))\\ = \int_{\hat{p}_2}^{\hat{p}_3} \frac{\partial W}{\partial p_3}(\hat{p}_1,\hat{p}_2,r)dr + \int_{\hat{p}_1}^{\hat{p}_2} \sum_{k=2}^{3} \frac{\partial W}{\partial p_k}(\hat{p}_1,r,r)dr + \int_{\hat{p}}^{\hat{p}_1} \sum_{k=1}^{3} \frac{\partial W}{\partial p_k}(r,r,r)dr.\end{aligned}$$

We know that $\hat{p}_1 < \hat{p}_2$ or $\hat{p}_2 < \hat{p}_3$. Assume first that $\hat{p}_2 < \hat{p}_3$. Then, we claim that the first integral in the right-hand side is strictly negative, while the two other ones are non-positive. Let us start with the first one. Let $r \in (\hat{p}_2, \hat{p}_3]$. Then,

$$\begin{aligned}\frac{\partial W}{\partial p_3}(\hat{p}_1,\hat{p}_2,r) &= \sum_{k=1}^{3} \left(\frac{\partial u}{\partial q_k} \frac{\partial D_k}{\partial p_3} - (c_u + c'(D_k)) \frac{\partial D_k}{\partial p_3} \right),\\ &= \frac{\partial D_3}{\partial p_3} \left(r - c_u - c'(D_3) \right) + \sum_{k=1}^{2} \frac{\partial D_k}{\partial p_3} \left(\hat{p}_k - c_u - c'(D_k) \right).\end{aligned}$$

Let $k \in \{1,2\}$. Firm 3 has the highest markup: $r - c_u - c'(D_3) > \hat{p}_k - c_u - c'(D_k)$. Indeed, since $r > \hat{p}_k$, $D_3 < D_k$, and, since costs are convex, $c'(D_3) < c'(D_k)$. Besides, firm 3's markup is strictly positive. Indeed, since $D_3(\hat{p}_1,\hat{p}_2,r) < \frac{1}{3}\sum_{k=1}^{3} D_k(\hat{p}_1,\hat{p}_2,r) \leq \frac{1}{3}\sum_{k=1}^{3} D_k(\hat{p},\hat{p},\hat{p}) = D_3(\hat{p},\hat{p},\hat{p})$, then $r - c_u - c'(D_3(\hat{p}_1,\hat{p}_2,r)) > \hat{p} - c_u - c'(D_3(\hat{p},\hat{p},\hat{p}))$, which is strictly positive, since $\pi(c_u) > 0$. As a result,

$$\frac{\partial W}{\partial p_3}(\hat{p}_1,\hat{p}_2,r) < (r - c_u - c'(D_3)) \sum_{k=1}^{3} \frac{\partial D_k}{\partial p_3} < 0.$$

Therefore, the first integral is indeed strictly negative, since $\hat{p}_2 < \hat{p}_3$. A similar argument shows that the two other integrands are non-positive, and we can conclude that the social welfare is strictly lower in a partial foreclosure equilibrium.

If $\hat{p}_1 < \hat{p}_2$, we can make the same reasoning to show that the first and third integrals are non-positive, while the second one is strictly negative. This concludes the proof. □